J. Fluid Mech. (1999), *vol.* 392, *pp.* 233–275. Printed in the United Kingdom

Nonlinear stability of a stratified shear flow in the regime with an unsteady critical layer. Part 2. Arbitrary stratification of asymmetric flow

By S. M. CHURILOV

Institute of Solar–Terrestrial Physics (ISTP), Siberian Department of Russian Academy of Sciences, Irkutsk 33, PO Box 4026, 664033, Russia

(Received 30 March 1998 and in revised form 20 February 1999)

A weakly nonlinear analysis of the downstream evolution of weakly unstable disturbances in a stably stratified mixing layer with a large Reynolds number is carried out. No other requirements are imposed upon velocity and density profiles, thus making it possible to overcome the restrictions placed in earlier studies (Brown & Stewartson 1978; Brown *et al.* 1981; Churilov & Shukhman 1987, 1988) by a particular choice of weakly supercritical flow models assuming symmetry. For each of the two critical layer regimes possible here, viscous and unsteady, evolution equations are obtained, their solutions and competition between nonlinearities in the course of instability development are analysed, and evolution scenarios for unstable disturbances are constructed for different levels of their supercriticality. It is established that the regime with a nonlinear critical layer does not arise in an evolutionary manner, except for the previously studied case of a weak stratification (Shukhman & Churilov 1997). It is shown that while in the viscous critical layer regime the relaxation of assumptions of the symmetry and weak supercriticality of the flow produces no fundamental changes in the theory, in the unsteady critical layer regime a new (non-dissipative cubic) nonlinearity appears which governs the instability development on equal terms with two already known nonlinearities. Results are illustrated by calculations for two families of flow models with a controlled degree of asymmetry.

1. Introduction

From the physical point of view, in free high-Reynolds-number shear flows the initiation of an instability and its development are both conditioned by the resonant wave–flow interaction (Andronov & Fabrikant 1979; Timofeev 1970, 1989) in the neighbourhood of a critical level $y = y_c$, i.e. the surface on which the flow velocity $v_x = u(y)$ coincides with the phase velocity of the wave c. Technically, the major research tool is weakly nonlinear theory which can work only at small amplitudes and near the stability boundary where growth rates are small, and an unstable disturbance may be represented as a slightly modified neutral mode. With small disturbance amplitudes and supercriticality, the resonance region, the so-called critical layer (CL) enclosing the surface $y = y_c$, is narrow. But it is here where the most interesting and important things happen, including the main nonlinear processes, and the narrowness of the CL plays, for theory, a key role.

Indeed, a disturbance of even a small amplitude is capable of dramatically restructuring the flow inside such a CL, and this, in turn, radically affects the course of its evolution. Such an interaction results in the formation of highly non-trivial evolution scenarios. Furthermore, because the main processes are localized inside the CL, nonlinear analysis is greatly simplified, and its results are nearly insensitive to the flow structure as a whole and, in this sense, universal, suitable for a broad class of flows; at the same time, however, they essentially depend on its local characteristics on a critical level.

With such a combination of the properties, the question of the robustness of the flow model chosen to be used in research, i.e. the question of how much results will be altered by a small variance of the model, becomes very important. This is especially true in regard to cases where the chosen model possesses some symmetry.

Since the CL plays a key role in the theory, it is important that the number of CLs must be a robust characteristic of the model. This requirement is satisfied in flows like a mixing layer where the velocity profile is monotonic and therefore there is a single CL. It is such flows that will be the subject of our treatment.

In the simplest case of a homogeneous mixing layer of incompressible fluid one could expect that the choice of the model (the velocity profile $u(y)$) should not be of substantial significance because the critical level necessarily coincides with the inflection point (maximum vorticity), $u_c'' \equiv u''(y_c) = 0$, and by Fjørtoft's theorem (see e.g. Drazin & Reid 1981), $u_c' u_c''' < 0$, i.e. the behaviour of $u(y)$ on a critical level is qualitatively the same for all flows which justifies to some extent the researchers' predilection for symmetric models in which $u(y) = u_0 + U(y)$ with $U(-y) = -U(y)$ (most often $U(y) = \tanh y$). Moreover, in all mixing layers, both with and without symmetry, the instability development proceeds qualitatively identically, differing only by minor details (see, for example, Goldstein & Hultgren 1988 and Churilov & Shukhman 1996). But the nonlinear evolution equation in the general case is structured in a manner quite different than in the particular case of a symmetric flow (cf. (2.34) and (2.32) in Churilov & Shukhman 1996); therefore, symmetric models are structurally unstable, and studying them alone can lead (and has led) to erroneous generalizations.

Another example is the weakly supercritical mixing layer $u = \tanh y$ on the β-plane (Churilov 1989). Here, the CL coincides with $\max u''(y)$ (i.e. $u_c''' = 0$, and $u_c^{\mathrm{iv}} < 0$) and is not at the symmetry centre of the flow model, owing to which even a symmetric model proves to be robust, and the evolution equation obtained for it becomes universal (see § 2.1 in Churilov & Shukhman 1996). In the case of the differential rotation of an incompressible fluid we have a similar situation (Shukhman 1989).

We now turn to the subject of this paper, namely stratified flows. Stratification, while being a very important factor for geophysical and astrophysical hydrodynamics, adds considerable complexity to the problem. Hence it comes as no surprise that, so far as is known to the author, all analytic studies on nonlinear stability of flows with finite continuous stratification employed symmetric models and, more commonly, Drazin's (1958) model

$$u = u_0 + \tanh y, \quad N^2(y) \equiv -g_0\rho_0'(y)/\rho_0(y) = J = \text{const}, \tag{1.1}$$

and Holmboe's (1960) model

$$u = u_0 + \tanh y, \quad N^2(y) = J/\cosh^2 y. \tag{1.2}$$

Here $\rho_0(y)$ is the density profile, g_0 is gravity acceleration, and the prime denotes the derivative with respect to the vertical coordinate y.

A distinguishing characteristic of such models is that the critical level lies in the symmetry plane, i.e. it coincides with the inflection point, as well as with extrema of $N^2(y)$ and of the Richardson number $R(y) = N^2(y)/[u'(y)]^2$. But, as shown by

Hazel (1972), in the general case, in the presence of a stratification the critical level ceases to be related to the maximum of vorticity (the inflection point), and also it does not coincide with extrema of $N^2(y)$ and $R(y)$. Thus, in a stratified mixing layer there is increasingly less reason to expect that the symmetric model will be robust compared with a homogeneous layer. It is also good to bear in mind that, even in symmetric models, only weakly supercritical flows have been studied to date, in which $R_c \equiv R(y_c) \approx \frac{1}{4}$, and instability is almost totally suppressed by stratification (Brown & Stewartson 1978; Brown, Rosen & Maslowe 1981; Churilov & Shukhman 1987, 1988). Another limiting case, a weakly stratified ($R_c \ll 1$) mixing layer, was also considered, but with an arbitrary velocity profile in this case (Shukhman & Churilov 1997).

The objective of this paper is to study the nonlinear evolution of weakly unstable disturbances in a stably stratified ($N^2 > 0$) mixing layer ($u' > 0$) virtually without imposing any other limitations on the velocity and density profile. This will make it possible to construct a unified theory for a wide variety of mixing layers with any degree of stratification, from weak to critical, and to see to what extent symmetric models represent correctly the general properties and what their demerits are.

It is clear that in the general case the spectrum of unstable disturbances will be wide (it is narrow in weakly supercritical flows only); therefore, it is necessary to have a mechanism for selecting weakly unstable modes from this wide spectrum. It will be assumed that the required disturbance is produced, as is often done in laboratory experiments, far upstream by an external source with a suitably chosen frequency and develops as it moves downstream (spatial evolution problem). Of course, the mechanism suggested and studied in detail in a homogeneous mixing layer by Goldstein & Hultgren (1988) and Hultgren (1992) appears to be more natural and attractive: the mode that initially has the largest growth rate reaches the stability boundary because of the viscous spreading of the velocity profile. In a stratified medium, however, the processes of dissipative spreading of the velocity and density profiles act upon the spectrum of unstable modes in a not so simple way, and such a problem statement must be preceded by a specially designed investigation.

In a stratified flow, the neutral mode is singular: when $y = y_c$ its linearized stream function $\psi = Ag(y)\,\mathrm{e}^{\mathrm{i}k(x-ct)}$ (with the amplitude A and the wavenumber k) has a branch point,

$$g(y) = (y - y_c)^{\alpha}[1 + a_1(y - y_c) + a_2(y - y_c)^2 + \cdots], \qquad \alpha(1 - \alpha) = R_c.$$

Hence the evolution scenario for unstable disturbances is fast (Churilov & Shukhman 1992): their development, depending on the amplitude and supercriticality (measured by the linear growth rate γ_L), proceeds in the regime of either viscous or unsteady CL (with an explosive growth up to $A = O(1)$ in the latter), and the nonlinear CL regime cannot be realized in the process of evolution (except for the case of a weak stratification $\alpha \ll 1$, Shukhman & Churilov 1997). It should be recalled that the CL will be viscous, unsteady or nonlinear according to which scale

$$l_v = v^{1/3}, \qquad l_t = |A|^{-1}\mathrm{d}|A|/\mathrm{d}s, \qquad l_N = |A|^{1/(2-\alpha)}, \tag{1.3}$$

viscous l_v, unsteady l_t or nonlinear l_N, is largest, and in the evolution process l_t and l_N vary, so transitions from one CL regime to another are possible. Here v is the reciprocal of the Reynolds number, and s is the evolution variable (time or coordinate downstream).

In the viscous CL regime, the instability development is described by the Landau–

Stuart–Watson equation

$$\frac{dA}{ds} = \gamma_L A + \frac{b_1(Pr-1)}{\nu^\sigma} R_c |A|^2 A, \quad \mathrm{Re}\, b_1 > 0, \tag{1.4}$$

with $\sigma = 1$ and $\mathrm{Im}\, b_1 = 0$ in the case of symmetric weakly supercritical ($R_c \approx \frac{1}{4}$) flows (Brown *et al.* 1981; Churilov & Shukhman 1987) and $\sigma = \frac{5}{3}$ in the case of a weak stratification ($\alpha \ll 1$, Shukhman & Churilov 1997). The evolution behaviour, as is easy to see, depends substantially on the Prandtl number Pr : when $Pr < 1$ the stabilization occurs at the level

$$|A| = A_1 = O[(\gamma_L \nu^\sigma / R_c)^{1/2}],$$

and when $Pr > 1$ we have an acceleration of the growth to an explosive one,

$$|A| \sim (s_1 - s)^{-1/2} \tag{1.5}$$

and the subsequent transition to the unsteady CL regime where the growth is also an explosive one

$$|A| \sim (s_0 - s)^{-a}, \quad a > 0, \tag{1.6}$$

but with different parameters.

It will be shown that in the general case the instability development in the viscous CL regime is also described by equation (1.4), and the presence or absence of the symmetry does not tangibly affect its structure, and a dependence of σ on α will be determined.

In the unsteady CL regime, the nonlinearity in the evolution equation has a characteristic non-local structure which was first found by Hickernell (1984). Churilov & Shukhman (1988, hereafter referred to as Paper I) showed that in weakly supercritical symmetric flows two main nonlinearities compete, namely the dissipative cubic nonlinearity which is a continuation of the nonlinearity in (1.4) into the domain of parameters corresponding to the unsteady CL regime, and the quintic in amplitude non-dissipative nonlinearity. Under the action of any one of them, the disturbance growth necessarily reaches the explosive stage (1.6), and only the parameters s_0 and a will be different. In the limit of a weak stratification ($R_c \ll 1$), of these two nonlinearities only a single, dissipative, nonlinearity survives. As will be shown, the reason is in different degree of their attenuation when $R_c \to 0$: the dissipative nonlinear term is proportional to R_c, and the non-dissipative term is proportional to R_c^2.

The main result of the relaxation of the flow symmetry assumption is the appearance of a third, non-dissipative cubic nonlinearity which controls the instability development on equal terms with the two above-mentioned nonlinearities. Its role is particularly important in the case of an intermediate stratification when $\alpha \leqslant \frac{1}{4}$ ($R_c \leqslant \frac{3}{16}$). This nonlinearity also accelerates the disturbance growth to an explosive one, and the respective term in the evolution equation has a similar non-local structure typical for the unsteady CL regime and is proportional to the parameter

$$Q = \left[\frac{u''}{u'} - \alpha \frac{(N^2)'}{N^2} \right]_{y=y_c}, \tag{1.7}$$

which in symmetric flows goes to zero.

The problem is solved by the method of matched asymptotic expansions (of solutions outside and inside the CL). In § 2, it is stated in detail, the necessary information from linear stability theory is given, the inner ($y \to y_c$) asymptotic expansion of the outer solution is calculated, and a modified solvability condition forming the basis for the future nonlinear evolution equation is derived. Three main (in the unsteady CL regime) nonlinear contributions to this equation are calculated in § 3; corresponding scalings are determined, and particular (for the individual nonlinearities) evolution equations are obtained, as well as the evolution equation for the viscous CL regime. Section 4 gives a description of the evolution of unstable disturbances in the unsteady CL regime, and results obtained are discussed in § 5.

In Appendix A, the solution of the equation basic to the inner problem is considered, and unwieldy kernels of the evolution equations are written in Appendix B.† Appendix C contains some details of calculations of the quintic nonlinearity. Finally, Appendix D is devoted to the study of some properties of the nonlinear evolution equations in the unsteady CL regime.

2. Problem statement and outer solution

2.1. *Selection of the model and input equations*

Let us consider plane-parallel free shear flows of stratified incompressible fluid with

$$v_x = u(y) \quad \text{and} \quad N^2(y) = Jr(y).$$

These are mixing-layer-type flows without counterflow (i.e. $u(y)$ is positive and monotonic; for definiteness, it will be assumed that $u'(y) > 0$), stably stratified ($r(y) > 0$), and the scale of variation of $r(y)$ is of the same order of magnitude or larger than the scale d of velocity variation. To exclude internal gravity waves from consideration, it is assumed that $r(y) \to 0$ when $y \to \pm\infty$. Otherwise $u(y)$ and $r(y)$ are arbitrary. All quantities are made dimensionless using the scale d, the flow velocity half-difference and the typical value of $|\rho_0'|$ as units.

The local Richardson number is

$$R(y) = Jr(y)/[u'(y)]^2. \tag{2.1}$$

Thus, the parameter $J > 0$ determines the degree of flow stratification, and by varying this parameter (with $u(y)$ and $r(y)$ fixed), it is possible to pass along the stability boundary from $R_c \approx 0$ to $R_c = \frac{1}{4}$. We are reminded that the prime denotes the derivative with respect to y, and the subscript c means that the quantity is calculated at $y = y_c$.

The dynamics of two-dimensional disturbances in the Boussinesq approximation is governed by the equations

$$\left.\begin{aligned} \frac{\partial}{\partial t}\Delta\psi + u\frac{\partial}{\partial x}\Delta\psi - u''\frac{\partial\psi}{\partial x} - J\frac{\partial\rho}{\partial x} + \{\Delta\psi,\psi\} &= v\Delta^2\psi, \\ \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + r\frac{\partial\psi}{\partial x} + \{\rho,\psi\} &= \frac{v}{Pr}\Delta\rho, \end{aligned}\right\} \tag{2.2}$$

where ψ and ρ are the pertubations of the stream function and density, v is the reciprocal of the Reynolds number, Pr is the Prandtl number, $\{a,b\} = (\partial a/\partial x)(\partial b/\partial y) - (\partial a/\partial y)(\partial b/\partial x)$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

We shall consider the spatial evolution downstream of weakly supercritical disturbances of a small amplitude produced at $x = -\infty$ by an external source with a frequency ω differing little from the neutral mode frequency ω_N which bounds (at a

† Appendix B is available from the *Journal of Fluid Mechanics* Editorial Office.

given J) the instability region:

$$|\omega - \omega_N| \ll \omega_N.$$

With such a problem statement, it is necessary to impose the boundary conditions

$$\psi \to 0, \quad \rho \to 0 \quad \text{as} \quad x \to -\infty. \tag{2.3}$$

2.2. *Linear theory*

In a linear approximation, $\psi \sim \exp[ik(x - ct)]$ and satisfies the Taylor–Goldstein equation:

$$\frac{d^2\psi}{dy^2} + \left[\frac{Jr}{(u-c)^2} - \frac{u''}{u-c} - k^2\right]\psi = 0; \qquad |\psi| < \infty \quad \text{as} \quad y \to \pm\infty, \tag{2.4}$$

where $k = \omega/c$, and the phase velocity $c = c(\omega, J)$ is the (complex in the general case) eigenvalue of the problem (2.4). Our interest is in the solutions of (2.4) on the stability boundary ($\mathrm{Im}\, c = 0^+$), the so-called neutral modes, as well as in its vicinity where disturbances are weakly unstable ($0 < \mathrm{Im}\, c \ll 1$). This subject is treated extensively in the literature (see, for example, Drazin & Reid 1981); among the references, of greatest utility to us are the publications of Miles (1961, 1963) who analysed in detail the neutral mode properties, Thorpe (1969) who constructed a broad class of analytic neutral solutions (2.4), and Hazel (1972) who studied the stability of asymmetric flows and their difference from symmetric flows.

In a homogeneous ($J = 0$) mixing layer, the modes are unstable in the interval $0 < \omega < \omega_0$. The eigenfunction $g(y)$ of the neutral mode corresponding to ω_0 (as well as the value of ω_0 itself) can be determined only by solving (2.4); it is known, however, that it has no zeros and is regular at a critical level, coincident with the inflection point $u_c'' = 0$. It is usual to normalize it with the condition $g(y_c) = 1$, so that

$$g(y) = 1 + a_1(y - y_c) + a_2(y - y_c)^2 + \cdots . \tag{2.5a}$$

On the other hand, when $\omega = 0$, in the general case y_c does not coincide with the inflection point, and the neutral mode has an ordinary zero:

$$g(y) \equiv (u - c)/u_c' = y - y_c + \frac{u_c''}{2u_c'}(y - y_c)^2 + \cdots; \quad c = \lim_{\omega \to 0} c(\omega). \tag{2.5b}$$

Stratification affects both the unstable mode spectrum and the neutral mode properties. Neutral modes become singular: when $y \to y_c$

$$\left.\begin{aligned} g(y) &= (y - y_c)^\alpha[1 + a_1(y - y_c) + \cdots], \quad \alpha(1 - \alpha) = R_c, \\ a_1 &= \frac{1}{2\alpha}\left[(1 + R_c)\frac{u_c''}{u_c'} - R_c\frac{r_c'}{r_c}\right] \end{aligned}\right\} \tag{2.6}$$

(because $u_c' > 0$, $g(y)$ should be continued analytically from $y > y_c$ into the lower half-plane of complex y). The stability boundary (neutral curve) on the (ω, J)-plane in the simplest cases has the form of an arc which connects the points (0, 0) and (ω_0, 0) bounding the instability region in a homogeneous flow, and the exponent α varies monotonically along it taking, in accordance with (2.5a, b), the values $\alpha = 1$ at the left-hand end and $\alpha = 0$ at the right-hand end (see figure 1). The instability region lies under the neutral curve.

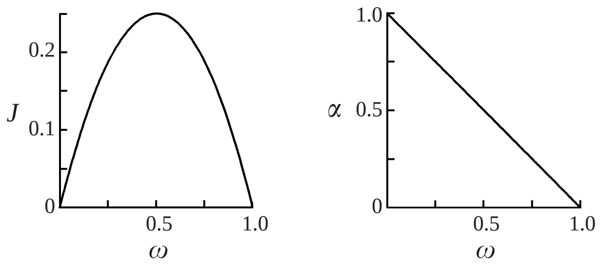

FIGURE 1. Stability boundary, and the $\alpha(\omega)$ dependence for Holmboe's model (1.2) with $u_0 = 1: \quad J = \omega(1-\omega), \quad \alpha = 1-\omega.$

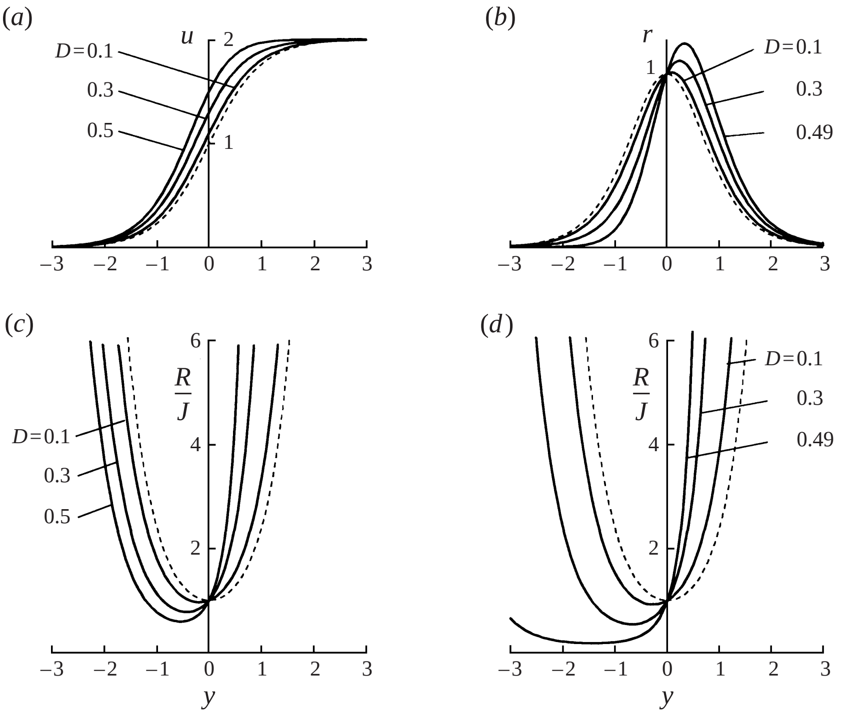

FIGURE 2. The arrangement of model flows (2.7a) and (2.7b) for different values of the asymmetry parameter D: (a) the velocity profile (the same for both models); (b) the density profile for (2.7b) (the dashed line is for (2.7a) and Holmboe's model); (c) and (d) the relative Richardson number profiles for (2.7a) and (2.7b) respectively (dashed lines are for Holmboe's model).

To gain greater insight into the influence of the flow asymmetry and have particular examples to illustrate results of the theory under development, we shall use two families of models:

$$u(y) = 1 + \tanh y + D/\cosh^2 y, \quad r(y) = \cosh^{-2} y \tag{2.7a}$$

and

$$u(y) = 1 + \tanh y + D/\cosh^2 y, \quad r(y) = (1 + 2D \tanh y)/\cosh^2 y. \tag{2.7b}$$

Their main features are shown in figure 2. The degree of asymmetry is controlled by the parameter D. When $D = 0$ both models transform to the symmetric Holmboe's

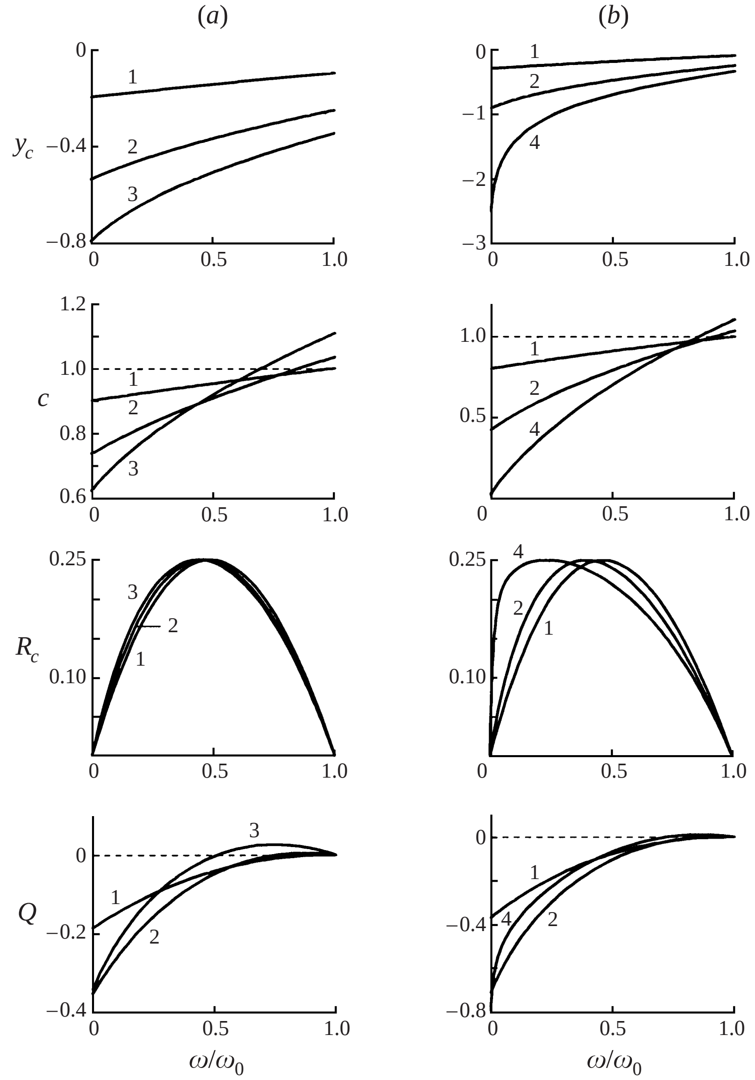

FIGURE 3. Variation of neutral mode parameters along the stability boundary: (*a*) model (2.7*a*), (*b*) model (2.7*b*). Curve 1, $D = 0.1$; 2, $D = 0.3$; 3, $D = 0.5$; 4, $D = 0.49$.

model (1.2) with $u_0 = 1$, where along the whole length of the stability boundary $c = u_0 = 1$, while the critical level y_c, the inflection point on the velocity profile, a maximum of $N^2(y)$ and a minimum of the Richardson number (see (2.1)) coincide and lie on $y = 0$. When D increases (in the range $|D| \leqslant \frac{1}{2}$ lest the monotonicity of $u(y)$ be violated), the inflection point shifts to the left, and a maximum of $r(y)$ in the model (2.7*b*) shifts to the right (figure 2). For $D \neq 0$ neutral modes and their parameters were calculated numerically. Figure 3 shows that the phase velocity c (and also y_c) varies not only with a change of D but also along the stability boundary.

With increasing asymmetry D, the stability boundary itself is distorted (see figure 4, dashes indicate points where R_c reaches the maximum value $\frac{1}{4}$) but its ‘height’ remains finite, so that for any $|D| \leqslant \frac{1}{2}$ there exists a value of the parameter $J =$

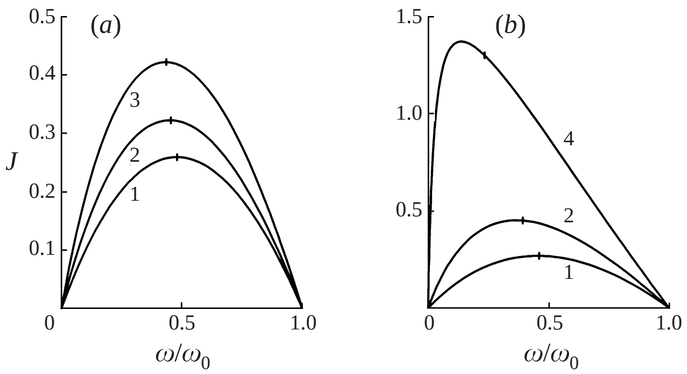

FIGURE 4. Stability boundary for different values of the parameter D: (a) model (2.7a), (b) model (2.7b). Curve 1, $D = 0.1$; 2, $D = 0.3$; 3, $D = 0.5$; 4, $D = 0.49$. The points where $R_c = \frac{1}{4}$ are marked by dashes.

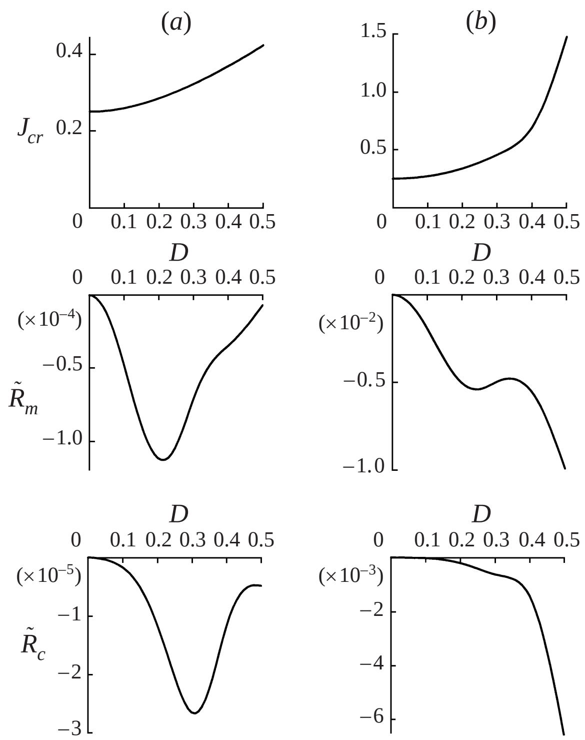

FIGURE 5. Marginally stable flows: J_{cr}, $\tilde{R}_m = \min R(y) - \frac{1}{4}$ and $\tilde{R}_c = R_c - \frac{1}{4}$ versus D: (a) model (2.7a), (b) model (2.7b).

$J_{cr} = \max_\omega J(\omega)$, at which the flow loses its stability. The symmetric Drazin's and Holmboe's models become unstable once an arbitrarily narrow region appears in the flow, where $R(y) < \frac{1}{4}$, and the first marginal mode has $R_c = \min R(y) = \frac{1}{4}$ (i.e. $\alpha = \frac{1}{2}$). Asymmetric flows, as established by Hazel (1972), even if they have a layer of a finite thickness with $R(y) < \frac{1}{4}$, can remain stable, and R_c of the first marginal mode does not coincide either with $\frac{1}{4}$ or with $R_m = \min R(y)$. Figure 5 shows, along with J_{cr} and R_m, R_c of the first marginal mode as function of D. It is seen that when $D \neq 0$ R_m and R_c are less than $\frac{1}{4}$ (it is interesting to note that $\alpha > \frac{1}{2}$ in this case); however, as in Hazel's (1972) calculations, the value of R_m does not drop below 0.2.

It should be noted that there are flows with a more complicated arrangement of the stability boundary where there are several neutral curves (see, for example, Miles 1963, as well as Drazin & Reid 1981), each of them not necessarily part of the stability boundary. Our intention here, however, does not involve analysing all possibilities – we need only have knowledge of the fact of the existence of the stability boundary and the presence of weakly unstable modes in its vicinity.

2.3. *Instability and outer solution*

By retreating slightly from the point (ω_N, J_N) on the neutral curve toward the instability region, we reach the weakly unstable mode whose development from a very small amplitude A right up to $A = O(1)$ can be investigated using perturbation theory. It is convenient to introduce the small parameters ε and μ characterizing the amplitude and supercriticality, as well as the designations

$$\xi = \mu x, \quad y - y_c = \mu Y, \quad \omega = \omega_N + \mu \Omega, \quad J = J_N + \mu J_1. \tag{2.8}$$

Recall that we will be studying largely the unsteady CL regime, which does specify the scaling of the inner variable Y; the inequality $\nu \ll \mu^3$ should also be satisfied. The relation between ε and μ depends upon which of the nonlinearities is the main one in the region of problem parameters considered; this issue will be taken up later in the text, in § 3.

The solution of equations (2.2) is constructed by the method of matched asymptotic expansions: first we find the solutions outside and inside the CL in the form of expansions in powers of ε and μ; after that, they are matched in each order. Matching gives a nonlinear evolution equation as the compatibility condition. Since this is a standard procedure, we examine briefly the main points only.

We construct the outer solution in the form of an expansion

$$\psi = [(\varepsilon\psi_1^{(1)} + \varepsilon\mu\psi_1^{(2)} + \varepsilon\nu\psi_1^{(3)})\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \varepsilon^2\psi_2^{(1)}\mathrm{e}^{2\mathrm{i}(kx-\omega t)} + \cdots + \text{c.c.}] + \varepsilon^2\psi_0^{(1)} + \cdots \tag{2.9}$$

and a corresponding expansion for ρ. Here the subscript and the superscript correspond, respectively, to the harmonic number and the iteration number, and c.c. represents the complex conjugates of the terms. The main part of the outer solution

$$\psi_1^{(1)} = A(\xi)g(y), \quad \rho_1^{(1)} = -\frac{r(y)}{u-c}\psi_1^{(1)},$$

is proportional to the neutral eigenfunction $g(y)$ of the (homogeneous) problem (2.4) with $\omega = \omega_N$ and $J = J_N$. The other expansion terms represent corrections for supercriticality (and the evolution caused by it), dissipation and nonlinearity. Being the general solution of a non-homogeneous equation, each of these terms includes the solution of the corresponding homogeneous equation with an arbitrary coefficient which will be denoted by b with appropriate indices. We do not show the calculations here (differing little from those done in Paper I and in many other publications) and give only the result for the inner (when $y \to y_c$) expansion of the fundamental (first) and second harmonics of the outer solution†:

$$\psi_1=\varepsilon\mu^{\alpha}\left[AY^{\alpha}-\frac{\alpha\dot{A}}{ku'_c}Y^{\alpha-1}+\cdots\right]+\varepsilon\mu^{\alpha+1}\left\{\frac{A}{2\alpha}\left[(1+R_c)\frac{u''_c}{u'_c}-R_c\frac{r'_c}{r_c}\right]Y^{\alpha+1}\right.$$
$$\left.+\left[\frac{R_c}{ku'_c}\dot{A}\left(\frac{r'_c}{r_c}-2\frac{u''_c}{u'_c}\right)+R_1A\right]\frac{Y^{\alpha}\ln Y}{1-2\alpha}\right\}+\varepsilon\nu\mu^{\alpha-3}\frac{\mathrm{i}R_c}{6\,Pr\,ku'_c}[(3-\alpha)Pr+1-\alpha]AY^{\alpha-3}$$
$$+\varepsilon\mu^{2-\alpha}\frac{f^{\pm}}{1-2\alpha}Y^{1-\alpha}+\lambda_1b_1^{\pm}Y^{\alpha}+\cdots, \tag{2.10}$$

$$\varphi_1=\varepsilon\mu^{\alpha+1}\left\{\frac{QA}{2\alpha}Y^{\alpha+1}-\left[(1-\alpha)(1+2\alpha)\frac{Q\dot{A}}{ku'_c}-2R_1A\right]\frac{Y^{\alpha}}{2\alpha(1-2\alpha)}\right\}$$
$$+\varepsilon\nu\mu^{\alpha-3}\frac{\mathrm{i}R_c(Pr-1)}{2Pr\,ku'_c}AY^{\alpha-3}+\varepsilon\mu^{2-\alpha}\frac{f^{\pm}}{1-\alpha}Y^{1-\alpha}+\cdots, \tag{2.11}$$

$$\psi_2=\varepsilon^2\mu^{2\alpha-2}\frac{A^2}{2(2-\alpha)u'_c}\left\{\alpha Y^{2\alpha-2}+\frac{\mu Y^{2\alpha-1}}{2-3\alpha}\left[(4-3\alpha-5\alpha^2+3\alpha^3)\frac{u''_c}{u'_c}-(4-3\alpha)R_c\frac{r'_c}{r_c}\right]\right\}$$
$$+\lambda_2b_2^{\pm}(\mu^{\alpha}q^{\pm}Y^{\alpha}+\mu^{1-\alpha}Y^{1-\alpha})+\cdots, \tag{2.12}$$

$$\varphi_2=-\varepsilon^2\mu^{2\alpha-1}\frac{\alpha QA^2}{2(2-\alpha)(2-3\alpha)u'_c}Y^{2\alpha-1}+\lambda_2\mu^{1-\alpha}\frac{1-2\alpha}{1-\alpha}b_2^{\pm}Y^{1-\alpha}+\cdots. \tag{2.13}$$

Here, instead of the density perturbation ρ, the function φ is used, which is more convenient for matching to the inner solution and is defined by

$$\varphi=\psi-\Pi,\quad \frac{\partial\Pi}{\partial y}=-\alpha\frac{u'_c}{r_c}\rho, \tag{2.14}$$

and the following designations are introduced:

$$\dot{A}=\mathrm{i}c\frac{\mathrm{d}A}{\mathrm{d}\xi}+\Omega A,\quad R_1=\frac{J_1r_c}{u_c'^{2}}\equiv\frac{J_1}{J_N}R_c. \tag{2.15}$$

The parameter $Q=u''_c/u'_c-\alpha r'_c/r_c$ (see (1.7) and figure 3) serves as some kind of flow asymmetry index and plays – as will be shown later in the text – an important role in nonlinear theory.

Arbitrary constants $b_{1,2}^{\pm}$ ($\pm$ indicates that $y\to y_c\pm 0$ accordingly) will be determined (together with the respective orders λ_1 and λ_2) by matching to the inner solution. The constants f^+ and f^- obey the modified solvability condition (MSC)

$$f^- - f^+=\frac{\dot{A}}{k}I_1-2\mathrm{i}k\frac{\mathrm{d}A}{\mathrm{d}\xi}I_2-J_1AI_3, \tag{2.16}$$

$$I_1=\int_{-\infty}^{\infty}\mathrm{d}y g^2(y)\left[\frac{u''}{(u-c)^2}-\frac{2Jr}{(u-c)^3}\right],\quad I_2=\int_{-\infty}^{\infty}\mathrm{d}y g^2(y),\quad I_3=\int_{-\infty}^{\infty}\mathrm{d}y\frac{r(y)g^2(y)}{(u-c)^2},$$

which, after determining $f^{\pm}$ from the inner solution, gives a nonlinear evolution equation. The integrals I_1 and I_3 should be evaluated by indenting the singular point

† We assume that Y^{α} is analytic in the lower half-plane of complex Y, and $\alpha\neq\frac{1}{2}$. Results for $\alpha=\frac{1}{2}$ can be obtained by taking the limit and are also contained in Paper I.

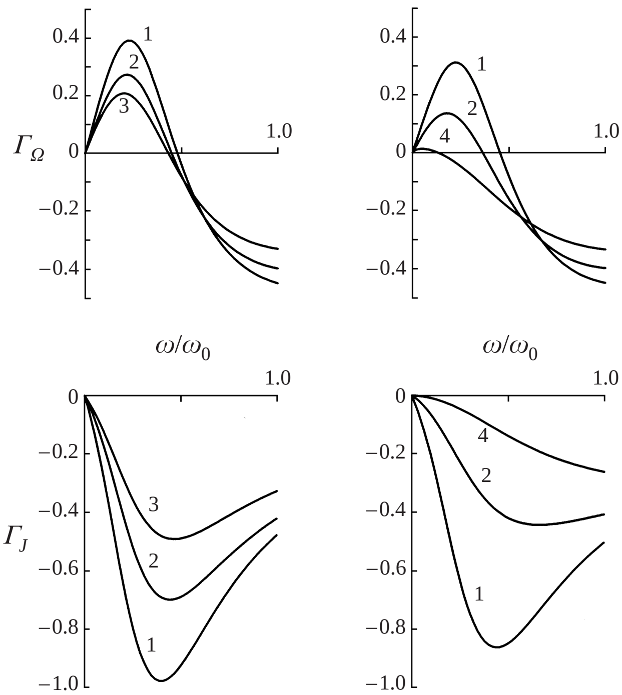

FIGURE 6. Variation of the 'growth rate components' Γ_Ω and Γ_J along the stability boundary: (*a*) model (2.7*a*), (*b*) model (2.7*b*). Curve 1, $D = 0.1$; 2, $D = 0.3$; 3, $D = 0.5$; 4, $D = 0.49$.

from below. It should be noted that the outer solution including the MSC (2.16) does not differ fundamentally from that obtained in Paper I.

In the linear approximation, $f^+ = f^-$, and (2.16) gives the equation

$$\frac{\mathrm{d}A}{\mathrm{d}\xi} = \frac{\mathrm{i}}{I_0}\left[I_1\Omega - kI_3J_1\right]A, \qquad I_0 = cI_1 - 2k^2I_2, \tag{2.17}$$

describing an exponential growth in amplitude at the initial stage of evolution:

$$A = A_0 \exp[(\Gamma + \mathrm{i}\kappa)\xi], \tag{2.18}$$

$$\kappa = \mathrm{Re}\,[(I_1\Omega - kI_3J_1)/I_0], \quad \Gamma = -\mathrm{Im}\,[(I_1\Omega - kI_3J_1)/I_0] \equiv \Gamma_\Omega\Omega + \Gamma_J J_1.$$

For the models (2.7*a*, *b*), Γ_Ω and Γ_J are plotted versus ω in figure 6. Note that Γ_Ω changes its sign in the point where $J(\omega)$ has its maximum whereas Γ_J remains negative everywhere because the instability region lies under the neutral curve.

3. Derivation of the nonlinear evolution equation

3.1. *Equations of the inner problem*

For constructing the solution inside the CL, we introduce, in addition to the inner coordinate, the variables Ψ and Φ (cf. (2.14)):

$$\psi = \varepsilon\mu^\alpha\Psi, \quad \alpha u_c'\rho/r_c = -\varepsilon\mu^{\alpha-1}P_Y, \quad \Phi = \Psi - P; \tag{3.1}$$

the subscript (Y, x or τ) implies differentiation with respect to the corresponding variable. Substituting into (2.2) and retaining, along with main-order terms, only the main corrections for nonlinearity, dissipation and unsteadiness we obtain the equations of the inner problem:

$$\hat{\mathrm{N}}^{(a)}\Psi - (1-\alpha)u_c'\Phi_x - \varepsilon\mu^{\alpha-2}(\Psi_{YY}\Psi_x - \Psi_{Yx}\Psi_Y) = \nu\mu^{-3}\Psi_{YYY}$$
$$+\mu\left\{u_c''Y\Psi_x - \frac{u_c''}{2}Y^2\Psi_{Yx} - \frac{u_c'}{c}\left[Y\Psi_{Y\tau} - \alpha\Psi_\tau - (1-\alpha)\Phi_\tau\right] - \frac{J_1}{\alpha u_c'}(\Psi_x - \Phi_x)\right\} + \cdots, \tag{3.2}$$

$$\hat{\mathrm{N}}^{(b)}\Phi - \varepsilon\mu^{\alpha-2}(\Phi_{YY}\Psi_x - \Phi_{Yx}\Psi_Y) = \frac{\nu\mu^{-3}}{Pr}[\Phi_{YYY} + \underbrace{(Pr-1)\Psi_{YYY}}_{\mathrm{I}}]$$
$$+\mu\left\{\underbrace{Q(u_c'Y - c_1)\Psi_x}_{\mathrm{II}} + \underbrace{j\Psi_x}_{\mathrm{III}} - \frac{u_c''}{2}Y^2\Phi_{Yx} - \frac{u_c'}{c}\left[Y\Phi_{Y\tau} - (1-\alpha)\Phi_\tau\right] + \frac{J_1}{\alpha u_c'}\Phi_x\right\} + \cdots, \tag{3.3}$$

where

$$\hat{\mathrm{N}}^{(a)} = \left[\frac{\partial}{\partial\tau} + (u_c'Y - c_1)\frac{\partial}{\partial x}\right]\frac{\partial}{\partial Y} - \alpha u_c'\frac{\partial}{\partial x},$$

$$\hat{\mathrm{N}}^{(b)} = \left[\frac{\partial}{\partial\tau} + (u_c'Y - c_1)\frac{\partial}{\partial x}\right]\frac{\partial}{\partial Y} - (1-\alpha)u_c'\frac{\partial}{\partial x}; \quad \tau = \frac{\xi}{c}, \quad c_1 = \frac{\Omega}{k}, \quad j = c_1Q - \frac{J_1}{\alpha u_c'}.$$

Equations (3.2) and (3.3) are analogous to equations (3.2) in Paper I. The difference lies in the fact that we are concerned with the spatial rather than temporal evolution, α in our case is arbitrary (in Paper I $\alpha = \frac{1}{2}$), and the flow possesses no symmetry ($u_c'' \neq 0$, $Q \neq 0$). The last difference is of the utmost significance because the additional nonlinearity (compared to Paper I) is associated with $Q \neq 0$.

By analogy with (2.9) we seek the solution of the inner problem in the form of a Fourier series expansion in x and each harmonic in the form of a series in powers of parameters $\varepsilon\mu^{\alpha-2}$, $\nu\mu^{-3}$ and μ small compared with unity. First, solving the equations (3.2) and (3.3) with zero right-hand sides, we find the main part of the inner solution. Then, taking into account the right-hand sides, we obtain dissipative ($\sim \nu\mu^{-3}$) and non-dissipative ($\sim \mu$) corrections as well as corrections of higher orders of smallness proportional to powers and products of these parameters. In calculating each iteration, we shall solve an equation of the form

$$\hat{\mathrm{N}}_l F_l \equiv \left\{\left[\frac{\partial}{\partial\tau} + \mathrm{i}lk(u_c'Y - c_1)\right]\frac{\partial}{\partial Y} - \mathrm{i}\beta lku_c'\right\}F_l = R_l$$

(the subscript indicates the harmonic number) considered in detail in Appendix A to reduce the mathematics in the main text to a minimum.

To close the MSC (2.16) and obtain the evolution equation, it is necessary to find $f^{\pm}$, i.e. the coefficients of the term $\sim Y^{1-\alpha}$ in the outer ($Y \to \pm\infty$) expansion of the fundamental harmonic ($l = 1$) of the inner solution. As follows from (2.11), such a term can appear if only $\Phi \neq 0$, and formulae (A 2), (A 6) and (A 8) explain this relation in terms of the inner solution and show that the non-trivial contribution ($f^+ \neq f^-$) is necessarily nonlinear.

There are three main contributions to Φ and therefore to the MSC (2.16). Two of them, dissipative and non-dissipative, are generated by the under-braced terms on the right-hand side of (3.3). As shown in Paper I, the third arises in the main part of the inner solution because the matching to the outer solution requires $\Phi \neq 0$. We shall calculate only the leading terms of these contributions, and it is convenient to divide all iterations of the inner solution necessary for this purpose into three sequences: the main sequence calculated at zero right-hand sides in (3.2) and (3.3), and two side sequences, dissipative ($\sim \nu\mu^{-3}$) and non-dissipative ($\sim \mu$). Accordingly, we seek the solution of the inner problem in the form of an expansion

$$\begin{aligned}\Psi_1 &= \big[\Psi_1^{(1)} + \varepsilon^2\mu^{2\alpha-4}\Psi_1^{(2)} + \varepsilon^4\mu^{2\alpha-7}\Psi_1^{(3)} + \nu\mu^{-3}\Psi_1^{(4)} + \varepsilon^2\nu\mu^{2\alpha-7}\Psi_1^{(5)} + \mu\Psi_1^{(6)} \\ &\quad + \varepsilon^2\mu^{2\alpha-3}\Psi_1^{(7)} + \cdots\big]\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \\ \Psi_0 &= \varepsilon\mu^{\alpha-2}\Psi_0^{(1)} + \varepsilon\nu\mu^{\alpha-5}\Psi_0^{(2)} + \varepsilon\mu^{\alpha-1}\Psi_0^{(3)} + \cdots, \\ \Psi_2 &= \big[\varepsilon\mu^{\alpha-2}\Psi_2^{(1)} + \varepsilon^3\mu^{3\alpha-6}\Psi_2^{(2)} + \varepsilon^3\mu^{\alpha-5}\Psi_2^{(3)} + \varepsilon\nu\mu^{\alpha-5}\Psi_2^{(4)} \\ &\quad + \varepsilon\mu^{\alpha-1}\Psi_2^{(5)} + \cdots\big]\mathrm{e}^{2\mathrm{i}(kx-\omega t)}, \\ \Psi_3 &= \varepsilon^2\mu^{2\alpha-4}\Psi_3^{(1)}\mathrm{e}^{3\mathrm{i}(kx-\omega t)} + \cdots,\end{aligned}$$

and, in a similar spirit, for Φ. The first to be written are the terms of the main sequence, followed by the dissipative and non-dissipative terms.

Finally, before starting the calculations we show that in the unsteady CL regime term III in (3.3) makes an excessively small, non-competitive contribution to (2.16). For this purpose, we truncate the equations (3.2) and (3.3) retaining only term III on their right-hand sides. The contribution to the solution of (3.3) linear in j (and fully nonlinear in A),

$$\Phi = \frac{\mu j}{(1-2\alpha)u_c'}\Psi_{\text{main}}, \tag{3.4}$$

is much smaller than Ψ_{main}, the main part of the inner solution, and contributes to (2.16) in the same way as Ψ_{main} does. Therefore, the contribution to (2.16) linear in j is merely a small correction to the contribution from the main sequence. In the next (quadratic in j) approximation, the contribution with the necessary asymptotic behaviour arises as early as in the first nonlinear ($\sim A^3$) iteration of the fundamental harmonic, $\Psi_1^{(j)} = O(\varepsilon^2\mu^{2\alpha-2})$, which, in view of (3.1), corresponds to $\psi_1^{(j)} = O(\varepsilon^3\mu^{3\alpha-2})$. In the unsteady CL regime

$$l_\nu \ll l_t \quad \text{and} \quad l_N \ll l_t, \quad \text{i.e.} \quad \nu \ll \mu^3 \quad \text{and} \quad \varepsilon \ll \mu^{2-\alpha}. \tag{3.5}$$

It is easy to see that for the matching of $\psi_1^{(j)}$ to (2.10) and for the competitiveness of the corresponding contribution to the MSC (2.16), an excessively large amplitude would be required, $\varepsilon = O(\mu^{2-2\alpha}) \gg \mu^{2-\alpha}$, unattainable in the unsteady CL regime; a further expansion in powers of j will give still smaller contributions.

3.2. *Quintic nonlinearity* (*contribution from the main sequence*)

3.2.1. $O(1)$ *of the fundamental*

The solution of the equations

$$\hat{\mathrm{N}}_1^{(a)}\Psi_1^{(1)} = \mathrm{i}(1-\alpha)ku_c'\Phi_1^{(1)}, \qquad \hat{\mathrm{N}}_1^{(b)}\Phi_1^{(1)} = 0,$$

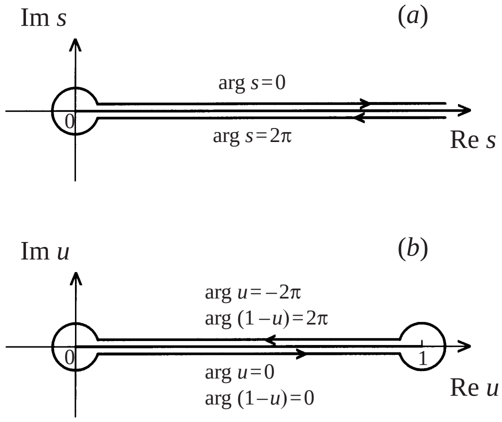

FIGURE 7. (*a*) Contour C, (*b*) contour L.

matching to (2.10) and (2.11) is readily obtained from (A 1) and (A 2):

$$\Psi_1^{(1)} \equiv W = \frac{\Gamma(\alpha+1)\exp[\mathrm{i}\pi(\alpha+1)/2]}{2\pi(ku_c')^{\alpha}}\int_C \mathrm{d}t\, t^{-\alpha-1}A(\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t}, \quad \Phi_1^{(1)}=0. \tag{3.6}$$

The contour C is shown in figure 7(*a*).

3.2.2. $O(\varepsilon\mu^{\alpha-2})$ *of the zeroth harmonic*

$$\Psi_{0Y\tau}^{(1)} = \mathrm{i}k\frac{\partial}{\partial Y}\left(\overline{W}'W-\overline{W}W'\right), \quad \Phi_{0Y\tau}^{(1)}=0. \tag{3.7}$$

Here, the prime denotes the derivative with respect to Y, and the overbar signifies a complex conjugate. Upon multiplying $\hat{\mathrm{N}}_1^{(a)}W=0$ by $\overline{W}'$ and adding to the complex conjugate expression, we obtain

$$\frac{\partial}{\partial\tau}|W'|^2 = \mathrm{i}\alpha ku_c'\left(\overline{W}'W-\overline{W}W'\right), \tag{3.8}$$

which permits us to integrate (3.7):

$$\Psi_0^{(1)}=\frac{|W'|^2}{\alpha u_c'}, \quad \Phi_0^{(1)}=0.$$

3.2.3. $O(\varepsilon\mu^{\alpha-2})$ *of the second harmonic*

$$\hat{\mathrm{N}}_2^{(a)}\Psi_2^{(1)} = \mathrm{i}k(W''W-W'^2)+2\mathrm{i}(1-\alpha)ku_c'\Phi_2^{(1)}, \quad \hat{\mathrm{N}}_2^{(b)}\Phi_2^{(1)}=0.$$

By means of (A 4) and (3.6) we find

$$\begin{aligned}\Psi_2^{(1)} = -\frac{k[\Gamma(\alpha+1)]^2\mathrm{e}^{\mathrm{i}\pi\alpha}}{8\pi^2(ku_c')^{2\alpha-1}}&\int_0^{\infty}\mathrm{d}s\int_C\mathrm{d}s_1\int_C\mathrm{d}s_2(s_1s_2)^{-\alpha-1}\\ &\times A(\tau-s-s_1)A(\tau-s-s_2)(s_1-s_2)^2\\ &\times(s_1+s_2)^{\alpha}(2s+s_1+s_2)^{-\alpha-1}\exp[-\mathrm{i}k(u_c'Y-c_1)(2s+s_1+s_2)],\end{aligned} \tag{3.9a}$$

$$\Phi_2^{(1)}=0.$$

By changing the variables, $t_1 = s + s_1$ and $t_2 = s + s_2$, and integrating over s, we get

$$\Psi_2^{(1)} = \frac{k\Gamma(\alpha)\Gamma(\alpha+1)e^{i\pi\alpha}}{4\pi^2(ku_c')^{2\alpha-1}} \int_C dt_1 \int_C dt_2 A(\tau-t_1)A(\tau-t_2)\left(\frac{t_1}{t_2}\right)^{\alpha} \frac{(t_1-t_2)^{1-\alpha}}{(t_1+t_2)^{1+\alpha}}$$
$$\times F\left(-\alpha, -\frac{\alpha}{2}; 1-\frac{\alpha}{2}; \frac{t_2^2}{t_1^2}\right) \exp[-ik(u_c'Y - c_1)(t_1+t_2)]\theta(t_1-t_2), \qquad (3.9b)$$

where $F(a, b, ; c; z)$ is a hypergeometric function (Abramowitz & Stegun 1964), and

$$\theta(z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0. \end{cases}$$

The function $\Psi_2^{(1)}$ is analytic in the lower half-plane (Im $Y < 0$) and is matched to the first term in (2.12).

3.2.4. $O(\varepsilon^2\mu^{2\alpha-4})$ *of the fundamental*

$$\hat{N}_1^{(a)}\Psi_1^{(2)} = ik[\Psi_{0YY}^{(1)}W - \Psi_{0Y}^{(1)}W' + 2\Psi_2^{(1)}\overline{W}'' - \Psi_{2Y}^{(1)}\overline{W}' - \Psi_{2YY}^{(1)}\overline{W}] + i(1-\alpha)ku_c'\Phi_1^{(2)},$$
$$\hat{N}_1^{(b)}\Phi_1^{(2)} = 0.$$

The solution is obtained by a standard method and has the form

$$\Psi_1^{(2)} = \frac{ik^2\Gamma(\alpha)[\Gamma(\alpha+1)]^2 e^{i\pi\alpha/2}}{8\pi^3(ku_c')^{3\alpha-2}} \int_0^{\infty} dt \int_C dt_1 \int_C dt_2 \int_{\overline{C}} dt_3 A(\tau-t-t_1)$$
$$\times A(\tau-t-t_2)\overline{A}(\tau-t-t_3)\frac{(t_1t_2t_3)^{-\alpha-1}(t_3-t_1-t_2)^{\alpha}}{(t_3-t-t_1-t_2)^{\alpha+1}}$$
$$\times\left\{t_1^{2\alpha+1}t_2\frac{(t_1-t_2)^{1-\alpha}}{(t_1+t_2)^{1+\alpha}}(t_1+t_2+t_3)(2t_3-t_1-t_2)F\left(-\alpha, -\frac{\alpha}{2}; 1-\frac{\alpha}{2}; \frac{t_2^2}{t_1^2}\right)\right.$$
$$\left. - t_3\left[t_1(t_1-t_3)^2 + t_2(t_2-t_3)^2 + t_1t_2(2t_3-t_1-t_2)\right]\right\}$$
$$\times\theta(t_1-t_2)\exp[ik(u_c'Y - c_1)(t_3 - t - t_1 - t_2)], \qquad (3.10)$$

$$\Phi_1^{(2)} = 0.$$

The contour $\overline{C}$ is complex conjugate with respect to the contour C. Note that, unlike W and $\Psi_2^{(1)}$, the functions $\Psi_0^{(1)}$ and $\Psi_1^{(2)}$ have singularities not only in the upper but also in the lower half-plane. As a result, for example, when $Y \to \pm\infty$

$$\Psi_1^{(2)} = D_1^{(2)\pm}Y^{\alpha} + O(Y^{\alpha-1}),$$

with $D_1^{(2)+} \neq D_1^{(2)-}$. Matching to (2.10), in view of (3.1), gives

$$\lambda_1 = \varepsilon^3\mu^{3\alpha-4}, \quad b_1^{\pm} = D_1^{(2)\pm}.$$

3.2.5. $O(\varepsilon^2\mu^{2\alpha-4})$ *of the third harmonic*

$$\hat{\mathrm{N}}_3^{(a)}\Psi_3^{(1)} = \mathrm{i}k(2\Psi_2^{(1)}W'' - 3\Psi_{2Y}^{(1)}W' + \Psi_{2YY}^{(1)}W) + 3\mathrm{i}(1-\alpha)ku_c'\Phi_3^{(1)},$$

$$\hat{\mathrm{N}}_3^{(b)}\Phi_3^{(1)} = 0.$$

By means of (3.6), (3.9*a*) and (A 4), we find $\Phi_3^{(1)} = 0$, and

$$\begin{aligned}\Psi_3^{(1)} = &-\frac{\mathrm{i}k^2[\Gamma(\alpha+1)]^3\mathrm{e}^{3\mathrm{i}\pi\alpha/2}}{(2\pi)^3(ku_c')^{3\alpha-2}}\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}t_4\int_C \mathrm{d}t_1\int_C \mathrm{d}t_2\int_C \mathrm{d}t_3\\ &\times A(\tau-t-t_1)A(\tau-t-t_2-t_4)A(\tau-t-t_3-t_4)\\ &\times\frac{(t_1t_2t_3)^{-\alpha-1}(t_2-t_3)^2(t_2+t_3)^\alpha(t_1+t_2+t_3+2t_4)^\alpha}{(t_2+t_3+2t_4)^{\alpha+1}(3t+t_1+t_2+t_3+2t_4)^{\alpha+1}}(t_1-t_2-t_3-2t_4)\\ &\times(2t_1-t_2-t_3-2t_4)\theta(t_2-t_3)\exp[-\mathrm{i}k(u_c'Y-c_1)(3t+t_1+t_2+t_3+2t_4)].\end{aligned} \quad (3.11)$$

3.2.6. $O(\varepsilon^3\mu^{3\alpha-6})$ *of the second harmonic*

$$\begin{aligned}\hat{\mathrm{N}}_2^{(a)}\Psi_2^{(2)} = \mathrm{i}k[&2\Psi_{0YY}^{(1)}\Psi_2^{(1)} - 2\Psi_{0Y}^{(1)}\Psi_{2Y}^{(1)} + \Psi_1^{(2)}W'' - 2\Psi_{1Y}^{(2)}W' + \Psi_{1YY}^{(2)}W + 3\Psi_3^{(1)}\overline{W}''\\ &-2\Psi_{3Y}^{(1)}\overline{W}' - \Psi_{3YY}^{(1)}\overline{W}] + 2\mathrm{i}(1-\alpha)ku_c'\Phi_2^{(2)},\end{aligned} \quad (3.12)$$

$$\hat{\mathrm{N}}_2^{(b)}\Phi_2^{(2)} = 0.$$

In Paper I, it is at this order that $\Phi \neq 0$ appeared. When $\alpha \neq \frac{1}{2}$, the situation is somewhat different because the last two parenthetical terms in the expansion (2.12) have different orders and $\Psi_2^{(2)}$ is matched only to the first of them, and $\Phi_2^{(2)} = 0$.

According to (A 6), when $Y \to \pm\infty$

$$\Psi_2^{(2)} = M^\pm Y^\alpha + O(Y^{\alpha-1}),$$

and $M^+ \neq M^-$ in view of the non-analyticity of the right-hand side of (3.12) in the lower half-plane (Im $Y < 0$). By means of simple but rather unwieldy calculations, it is possible to find (cf. (3.30) in Paper I)

$$\begin{aligned}M^+ - M^- = &\frac{2^{\alpha+1}\pi\alpha k^3\mathrm{e}^{\mathrm{i}\pi(1-\alpha)/2}}{\Gamma(1+\alpha)[\Gamma(1-\alpha)]^4(ku_c')^{3\alpha-3}}\int_0^\infty \mathrm{d}t\, t^{5-3\alpha}\int_0^1 \mathrm{d}x\, x\int_0^1 \mathrm{d}y\\ &\times G(x,y)A(\tau-t)A(\tau-xt)A(\tau-xyt)\overline{A}(\tau-(1+x+xy)t).\end{aligned} \quad (3.13)$$

The kernel $G(x, y)$ is given in Appendix B.

The matching $\Psi_2^{(2)}$ to (2.12), in view of (3.1), yields

$$\lambda_2 = \varepsilon^4\mu^{3\alpha-6}, \quad b_2^+q^+ = M^+, \quad b_2^-q^- = M^-. \quad (3.14)$$

But each of the expansions (2.12) and (more importantly) (2.13) contains yet another term $\sim \lambda_2 b_2^\pm\mu^{1-\alpha}Y^{1-\alpha} = \varepsilon^4\mu^{2\alpha-5}b_2^\pm Y^{1-\alpha}$, and these terms have no counterpart in the already constructed inner solution. For matching to them, it is necessary to add to the inner solution a corresponding contribution which cannot appear as a result of the generation of harmonics.

3.2.7. $O(\varepsilon^3\mu^{\alpha-5})$ *of the second harmonic*

$$\hat{\mathrm{N}}_2^{(a)}\Psi_2^{(3)} = 2\mathrm{i}(1-\alpha)ku_c'\Phi_2^{(3)}, \quad \hat{\mathrm{N}}_2^{(b)}\Phi_2^{(3)} = 0.$$

The solution of these homogeneous equations is easily obtained using (A 1) and (A 8):

$$\Phi_2^{(3)} = -\frac{\Gamma(2-\alpha)\mathrm{e}^{-\mathrm{i}\pi\alpha/2}}{2\pi(2ku_c')^{1-\alpha}}\int_C \mathrm{d}t\, t^{\alpha-2}C(\tau-t)\mathrm{e}^{-2\mathrm{i}k(u_c'Y-c_1)t}, \quad \Psi_2^{(3)} = \frac{1-\alpha}{1-2\alpha}\Phi_2^{(3)}.$$

The matching to (2.12) and (2.13) gives (in view of (3.14))

$$b_2^+ = b_2^- = \frac{M^+-M^-}{q^+-q^-}, \quad C(\tau) = \frac{M^+-M^-}{(1-\alpha)\varDelta}, \quad \varDelta = \frac{q^+-q^-}{1-2\alpha}, \tag{3.15}$$

In the general case q^+ and q^- are finite and different, but other variants are possible; they are all treated in Appendix C.

It should be emphasized that the mechanism for generating Φ in the process of matching to the outer solution is such that it changes the order of the corresponding (second in this case) harmonic by a factor $\mu^{1-2\alpha}$. As a result, in the case of the ordinary generation of harmonics successive contributions to the same harmonic (for example, $\Psi_1^{(1)}$ and $\Psi_1^{(2)}$) differ by the factor $F_1 = \varepsilon^2\mu^{2\alpha-4} \ll 1$ (see (3.5)), and in the case of the matching-induced generation of Φ they differ by the factor

$$F_0 = \varepsilon^2\mu^{-3} \ll 1. \tag{3.16}$$

The last inequality is indispensable for the correctness of the perturbation theory being developed here.

3.2.8. $O(\varepsilon^4\mu^{2\alpha-7})$ *of the fundamental*

$$\hat{\mathrm{N}}_1^{(a)}\Psi_1^{(3)} = \mathrm{i}k(2\Psi_2^{(3)}\overline{W}'' - \Psi_{2Y}^{(3)}\overline{W}' - \Psi_{2YY}^{(3)}\overline{W}) + \mathrm{i}(1-\alpha)ku_c'\Phi_1^{(3)},$$
$$\hat{\mathrm{N}}_1^{(b)}\Phi_1^{(3)} = -\mathrm{i}k\left(2\Phi_{2Y}^{(3)}\overline{W}' + \Phi_{2YY}^{(3)}\overline{W}\right).$$

It is evident that at this order the solution has the required asymptotic behaviour

$$\Phi_1^{(3)} = m^\pm Y^{1-\alpha} + O(Y^{-\alpha}), \quad \Psi_1^{(3)} = \frac{1-\alpha}{1-2\alpha}m^\pm Y^{1-\alpha} + n^\pm Y^\alpha + O(Y^{\alpha-1}+Y^{-\alpha}),$$

and the matching to (2.10) and (2.11) calls for a scaling

$$\varepsilon^4 = \mu^{9-4\alpha}. \tag{3.17}$$

By means of (A 6), we find

$$m^+ - m^- = -\frac{4\mathrm{i}k\alpha\pi^{3/2}(ku_c')^{1-\alpha}\mathrm{e}^{\mathrm{i}\pi\alpha/2}}{\Gamma(1-\alpha)\Gamma\left(\frac{1}{2}(1-\alpha)\right)\Gamma\left(\frac{1}{2}(2+\alpha)\right)}\int_0^\infty \mathrm{d}t\, t^{1-\alpha}C(\tau-t)\overline{A}(\tau-2t).$$

In view of (3.15) and (3.13), with a little rearrangement, we may write

$$m^+ - m^- = \frac{2^{\alpha+2}\alpha^2\pi^{5/2}k^4(ku_c')^{4-4\alpha}}{\Gamma(1+\alpha)[\Gamma(1-\alpha)]^5\Gamma\left(\frac{1}{2}(2+\alpha)\right)\Gamma\left(\frac{1}{2}(3-\alpha)\right)\varDelta}I^{(5)}, \tag{3.18}$$

where

$$I^{(5)} = \int_0^\infty \mathrm{d}t\, t^{1-\alpha} \int_0^\infty \mathrm{d}s\, s^{5-3\alpha} \int_0^1 \mathrm{d}x\, x \int_0^1 \mathrm{d}y\, G(x,y)$$
$$\times A(\tau - t - s)A(\tau - t - xs)A(\tau - t - xys)$$
$$\times \overline{A}(\tau - t - (1 + x + xy)s)\overline{A}(\tau - 2t)$$
$$= \int_0^\infty \mathrm{d}t\, t^{7-4\alpha} \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, H(s,y,z)A(\tau - t)A(\tau - st)A(\tau - syt)$$
$$\times \overline{A}(\tau - 2syzt)\overline{A}(\tau - (1 + s + sy - 2syz)t),$$
$$H(s,y,z) = s^{3-\alpha}y^{2-\alpha}z^{1-\alpha}(1 - syz)^{3-3\alpha}G\left(s\frac{1-yz}{1-syz},\ y\frac{1-z}{1-yz}\right).$$

This is just the contribution to the MSC (2.16) from the main sequence.

Before proceeding to calculating the contributions of the side sequences, we should note that the structure of the nonlinear term in (3.3) is such that Ψ of the main sequence and Φ of the side sequence participate in the generation of harmonics. It will therefore suffice to calculate Φ only.

3.3. *Dissipative nonlinearity*

This nonlinearity is caused by the contribution to Φ which is due to the dissipative term I on the right-hand side of (3.3).

3.3.1. $O(\nu\mu^{-3})$ *of the fundamental*

$$\hat{\mathrm{N}}_1^{(a)}\Psi_1^{(4)} = W''' + \mathrm{i}(1-\alpha)ku_c'\Phi_1^{(4)}, \quad \hat{\mathrm{N}}_1^{(b)}\Phi_1^{(4)} = \frac{Pr-1}{Pr}W'''.$$

Since

$$\frac{\partial^3}{\partial Y^3}\hat{\mathrm{N}}_1^{(a)}W = (\hat{\mathrm{N}}_1^{(a)} + 3\mathrm{i}ku_c')W''' = \left[\hat{\mathrm{N}}_1^{(b)} + \mathrm{i}(4-2\alpha)ku_c'\right]W''' = 0,$$

then

$$\Phi_1^{(4)} = \frac{\mathrm{i}(Pr-1)}{2(2-\alpha)Pr\,ku_c'}W''', \quad \Psi_1^{(4)} = \mathrm{i}\frac{(3-\alpha)Pr + 1 - \alpha}{6(2-\alpha)Pr\,ku_c'}W'''. \tag{3.19}$$

3.3.2. $O(\varepsilon\nu\mu^{\alpha-5})$ *of the zeroth harmonic*

$$\Phi_{0\tau Y}^{(2)} = \mathrm{i}k\frac{\partial}{\partial Y}\left(\overline{\Phi_{1Y}^{(4)}}W - \Phi_{1Y}^{(4)}\overline{W}\right) + \frac{Pr-1}{Pr}\Psi_{0YYY}^{(1)}.$$

Upon integrating with respect to Y and τ, in view of the fact that $\Phi \to 0$ when $\tau \to -\infty$, we obtain

$$\Phi_0^{(2)} = \frac{(Pr-1)k\Gamma(\alpha)\Gamma(\alpha+1)}{4\pi^2 Pr(ku_c')^{2\alpha-3}} \int_0^\infty \mathrm{d}t \int_C \mathrm{d}t_1 \int_{\overline{C}} \mathrm{d}t_2$$
$$\times A(\tau - t - t_1)\overline{A}(\tau - t - t_2)(t_1t_2)^{-\alpha-1}$$
$$\times \left[\frac{\alpha}{2(2-\alpha)}(t_1^4 + t_2^4) - t_1t_2(t_1 - t_2)^2\right]\exp[-\mathrm{i}k(u_c'Y - c_1)(t_1 - t_2)].$$

3.3.3. *$O(\varepsilon\nu\mu^{\alpha-5})$ of the second harmonic*

$$\hat{\mathrm{N}}_2^{(b)}\Phi_2^{(4)} = \mathrm{i}k\left(\Phi_{1YY}^{(4)}W - \Phi_{1Y}^{(4)}W'\right) + \frac{Pr-1}{Pr}\Psi_{2YYY}^{(1)}.$$

By means of (3.6), (3.9*b*) and (3.19), we get

$$\Phi_2^{(4)} = \frac{(Pr-1)k\Gamma(\alpha)\Gamma(\alpha+1)}{4\pi^2 Pr(ku_c')^{2\alpha-3}}\mathrm{e}^{\mathrm{i}\pi\alpha}\int_0^\infty \mathrm{d}t\int_C \mathrm{d}t_1\int_C \mathrm{d}t_2 A(\tau-t-t_1)A(\tau-t-t_2)$$
$$\times\left[\frac{\alpha}{2(2-\alpha)}(t_1-t_2)^2(t_1^2+t_2^2) - t_1^{2\alpha+1}t_2(t_1^2-t_2^2)^{1-\alpha}F\left(-\alpha,-\frac{\alpha}{2};1-\frac{\alpha}{2};\frac{t_2^2}{t_1^2}\right)\right]$$
$$\times\left(\frac{t_1+t_2}{2t+t_1+t_2}\right)^{2-\alpha}(t_1t_2)^{-\alpha-1}\theta(t_1-t_2)\exp[-\mathrm{i}k(u_c'Y-c_1)(2t+t_1+t_2)].$$

3.3.4. *$O(\varepsilon^2\nu\mu^{2\alpha-7})$ of the fundamental*

$$\hat{\mathrm{N}}_1^{(b)}\Phi_1^{(5)} = \mathrm{i}k[\Phi_{0YY}^{(2)}W - \Psi_{0Y}^{(1)}\Phi_{1Y}^{(4)} + 2\Psi_2^{(1)}\overline{\Phi_{1YY}^{(4)}} + \Psi_{2Y}^{(1)}\overline{\Phi_{1Y}^{(4)}} - \Phi_{2YY}^{(4)}\overline{W} - 2\Phi_{2Y}^{(4)}\overline{W}']$$
$$+\frac{Pr-1}{Pr}\Psi_{1YYY}^{(2)}.$$

According to (A 6), when $Y\to\pm\infty$

$$\Phi_1^{(5)} = m_\nu^{\pm}Y^{1-\alpha} + O(Y^{-\alpha}),$$

where

$$m_\nu^+ - m_\nu^- = -\frac{\pi(Pr-1)\alpha(1-\alpha)^2k^2\mathrm{e}^{-\mathrm{i}\pi\alpha}}{Pr\Gamma(1-\alpha)\Gamma(3-\alpha)(ku_c')^{4\alpha-5}}\int_0^\infty \mathrm{d}t\, t^{7-4\alpha}\int_0^1 \mathrm{d}\sigma\,\sigma^{3-2\alpha}G_\nu(\sigma)$$
$$\times A(\tau-t)A(\tau-\sigma t)\overline{A}(\tau-(1+\sigma)t). \tag{3.20}$$

The index ν signifies the dissipative nature of the contribution. The kernel $G_\nu(\sigma)$ has alternating signs (see figure 8*a*). In explicit form it is written in Appendix B.

Matching $\Phi_1^{(5)}$ to (2.11), in view of (3.1), calls for the scaling

$$\varepsilon^2\nu = \mu^{9-4\alpha}. \tag{3.21}$$

3.4. *Non-dissipative cubic nonlinearity*

This nonlinearity is caused by the contribution to Φ which is due to the non-dissipative term II on the right-hand side of (3.3).

3.4.1. *$O(\mu)$ of the fundamental*

$$\hat{\mathrm{N}}_1^{(b)}\Phi_1^{(6)} = \mathrm{i}kQ(u_c'Y-c_1)W. \tag{3.22}$$

Let us introduce the function

$$V = -\frac{\Gamma(\alpha+1)\mathrm{e}^{\mathrm{i}\pi\alpha/2}}{2\pi(ku_c')^{\alpha+1}}\int_C \mathrm{d}t\, t^{-\alpha-2}A(\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t},$$

which satisfies the relation $V_Y = W$ and the equation

$$(\hat{\mathrm{N}}_1^{(a)} - \mathrm{i}ku_c')V = 0.$$

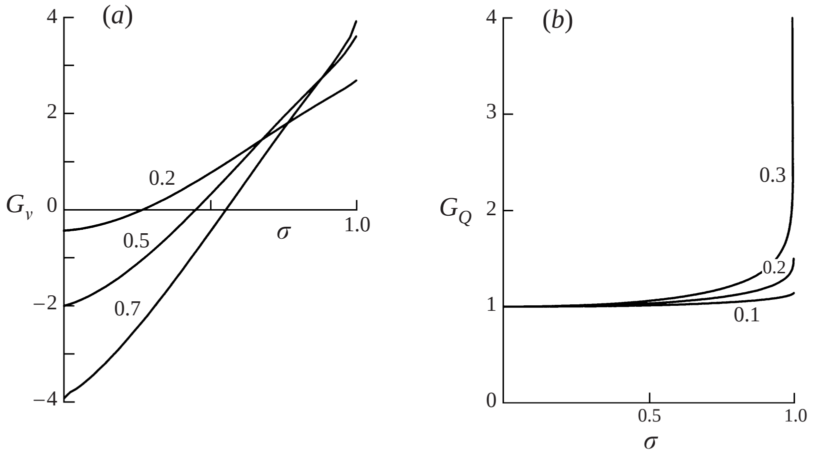

FIGURE 8. Kernels of cubic nonlinear terms versus σ: (*a*) dissipative nonlinearity, (*b*) non-dissipative non-linearity; each curve is marked by the corresponding value of α.

It is an easy matter to show that the solution of (3.22) has the form

$$\Phi_1^{(6)} = \frac{Q}{2\alpha(1-2\alpha)}\left[(1+\alpha)V - \frac{2\alpha}{u_c'}(u_c'Y - c_1)W\right].$$

3.4.2. $O(\varepsilon\mu^{\alpha-1})$ *of the zeroth harmonic*

$$\Phi_{0\tau Y}^{(3)} = \mathrm{i}k\frac{\partial}{\partial Y}\left(\overline{\Phi_{1Y}^{(6)}}W - \Phi_{1Y}^{(6)}\overline{W}\right) = \frac{\mathrm{i}kQ}{(1-2\alpha)u_c'}\frac{\partial}{\partial Y}\left[(u_c'Y - c_1)(W'\overline{W} - W\overline{W}')\right].$$

By means of (3.8), we obtain

$$\Phi_0^{(3)} = -\frac{Q(u_c'Y - c_1)|W'|^2}{\alpha(1-2\alpha)u_c'^2}.$$

3.4.3. $O(\varepsilon\mu^{\alpha-1})$ *of the second harmonic*

$$\hat{\mathrm{N}}_2^{(b)}\Phi_2^{(5)} = \mathrm{i}k\left(\Phi_{1YY}^{(6)}W - \Phi_{1Y}^{(6)}W'\right) + 2\mathrm{i}kQ(u_c'Y - c_1)\Psi_2^{(1)}$$

$$= \mathrm{i}kQ\left\{2(u_c'Y - c_1)\Psi_2^{(1)} - \frac{1}{(1-2\alpha)u_c'}\left[(u_c'Y - c_1)(W''W - W'^2) + u_c'WW'\right]\right\}.$$

The solution of this equation has the form

$$\Phi_2^{(5)} = -\frac{\mathrm{i}kQ\Gamma(\alpha)\Gamma(\alpha+1)\mathrm{e}^{\mathrm{i}\pi\alpha}}{8\pi^2(1-2\alpha)(ku_c')^{2\alpha}}\int_0^\infty \mathrm{d}t\int_C \mathrm{d}t_1\int_C \mathrm{d}t_2 A(\tau - t - t_1)A(\tau - t - t_2)\frac{(t_1t_2)^{-\alpha-1}}{(2t+t_1+t_2)^{2-\alpha}}$$
$$\times\left\{\frac{(t_1-t_2)^2(t_1+t_2)^\alpha}{(2t+t_1+t_2)^{2\alpha}}[1+\alpha+2\mathrm{i}k\alpha(u_c'Y - c_1)(2t+t_1+t_2)]\right.$$
$$\left.-\frac{(1-\alpha)(t_1-t_2)^2 + 2\alpha(t_1+t_2)^2}{(t_1+t_2)^\alpha}\right\}\theta(t_1-t_2)\exp[-\mathrm{i}k(u_c'Y - c_1)(2t+t_1+t_2)].$$

3.4.4. $O(\varepsilon^2\mu^{2\alpha-3})$ *of the fundamental*

$$\hat{\mathrm{N}}_1^{(b)}\Phi_1^{(7)}=\mathrm{i}k\big[\Phi_{0YY}^{(3)}W-\Psi_{0Y}^{(1)}\Phi_{1Y}^{(6)}+2\Psi_2^{(1)}\overline{\Phi_{1YY}^{(6)}}+\Psi_{2Y}^{(1)}\overline{\Phi_{1Y}^{(6)}}-\Phi_{2YY}^{(5)}\overline{W}-2\Phi_{2Y}^{(5)}\overline{W'}\big]$$
$$+\mathrm{i}kQ(u_c'Y-c_1)\Psi_1^{(2)}.$$

According to (A 6), when $Y\to\pm\infty$

$$\Phi_1^{(7)}=m_Q^{\pm}Y^{1-\alpha}+O(Y^{-\alpha}),$$

where

$$m_Q^+-m_Q^-=\frac{2\mathrm{i}\pi k^2Q(1-\alpha)(ku_c')^{2-4\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{(1-2\alpha)[\Gamma(1-\alpha)]^2\Gamma(2-2\alpha)}\int_0^\infty \mathrm{d}t\,t^{3-4\alpha}$$
$$\times\int_0^1\mathrm{d}\sigma\,\sigma^{2-2\alpha}G_Q(\sigma)A(\tau-t)A(\tau-\sigma t)\overline{A}(\tau-(1+\sigma)t). \quad (3.23)$$

The kernel $G_Q(\sigma)$ is a positive monotonically growing function of σ (see figure 8*b*). In explicit form it is written in Appendix B. The matching of $\Phi_1^{(7)}$ to (2.11) calls for the scaling

$$\varepsilon^2=\mu^{5-4\alpha}. \quad (3.24)$$

Noteworthy is the fact that in this Subsection all iterations of Φ, and $m_Q^+-m_Q^-$ along with them, contain the factor $(1-2\alpha)^{-1}$ and tend to infinity when $\alpha\to\frac{1}{2}$. If, however, calculations are performed directly for $\alpha=\frac{1}{2}$, it turns out that all quantities are finite, but the integrals expressing them contain not only the powers of t_i but also logarithms. To reconcile these results with those obtained above, it is necessary to construct for $\alpha\to\frac{1}{2}$ an expansion of the solution (3.3) differing from that described above in that it takes into account explicitly both the smallness of $(1-2\alpha)$ and contributions to the complete solution of the form (3.4). We do not reproduce these calculations here because the non-dissipative cubic nonlinearity, as will be shown in § 4, influences the instability development significantly only if $\alpha\leqslant\frac{1}{4}$ when (3.23) holds.

3.5. *Nonlinear evolution equations in the unsteady CL regime*

Thus, in the unsteady CL regime there are three main contributions to the MSC (2.16) from the inner solution: dissipative, non-dissipative cubic, and non-dissipative quintic. We have calculated the first terms of the expansions of these contributions (respectively, (3.20), (3.23) and (3.18)) in parameters $F_0=\varepsilon^2\mu^{-3}$ and $F_1=\varepsilon^2\mu^{2\alpha-4}$ which were taken to be small, and we have now to substitute them into (2.16) to obtain the nonlinear evolution equation (NEE). To each contribution, however, there corresponds its own scaling ((3.21), (3.24) and (3.17), respectively), and these scalings can be carried out simultaneously only when $\alpha=\frac{1}{4}$ and the amplitude and supercriticality (γ_L) are such that $\varepsilon=\nu^{1/2}$ and $\mu=\nu^{1/4}$. This is just a point in the space of problem parameters, and only at this point do the three nonlinearities have the same order of magnitude. In all the remaining space of parameters, the nonlinearities have different orders of magnitude, and only one of them determines the instability development; in this region that is the main one, and the other two are of secondary importance. Therefore, first we substitute into the MSC (2.16) the individual nonlinear contributions of the CL to obtain and analyse three NEEs, and only after that do we bring together the three nonlinearities in a single NEE and study their competition.

Taking into consideration that when matching Φ to (2.11)

$$f^+ - f^- = (1-\alpha)(m^+ - m^-),$$

passing to 'physical' variables (see (2.8))

$$x = c\tau/\mu, \quad \gamma_L = \mu\Gamma, \quad \tilde{A} = \varepsilon A \mathrm{e}^{-\mathrm{i}\kappa\xi}$$

and then dropping the tilde above A, we obtain the following NEEs:
with the dissipative nonlinearity

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + \frac{Pr-1}{Pr} \nu b_1 \mathrm{e}^{-\mathrm{i}\chi_1} \int_0^\infty \mathrm{d}s\, s^{7-4\alpha} \int_0^1 \mathrm{d}\sigma\, \sigma^{3-2\alpha} \times G_\nu(\sigma) A(x-s) A(x-\sigma s) \overline{A}(x-(1+\sigma)s); \tag{3.25}$$

with the non-dissipative cubic nonlinearity

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A - \mathrm{i}Q b_2 \mathrm{e}^{-\mathrm{i}\chi_1} \int_0^\infty \mathrm{d}s\, s^{3-4\alpha} \int_0^1 \mathrm{d}\sigma\, \sigma^{2-2\alpha} \times G_Q(\sigma) A(x-s) A(x-\sigma s) \overline{A}(x-(1+\sigma)s); \tag{3.26}$$

with the non-dissipative quintic nonlinearity

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} &= \gamma_L A + b_3 \mathrm{e}^{-\mathrm{i}\chi_2} \int_0^\infty \mathrm{d}t\, t^{1-\alpha} \int_0^\infty \mathrm{d}s\, s^{5-3\alpha} \int_0^1 \mathrm{d}v\, v \int_0^1 \mathrm{d}w \\ &\quad \times G(v,w) A(x-t-s) A(x-t-vs) A(x-t-vws) \overline{A}(x-t-(1+v+vw)s) \overline{A}(x-2t) \\ &= \gamma_L A + b_3 \mathrm{e}^{-\mathrm{i}\chi_2} \int_0^\infty \mathrm{d}s\, s^{7-4\alpha} \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}w\, H(t,v,w) A(x-s) A(x-ts) A(x-tvs) \\ &\quad \times \overline{A}(x-2tvws) \overline{A}(x-(1+t+tv-2tvw)s); \end{aligned} \tag{3.27}$$

where

$$b_1 = \frac{\pi\alpha(1-\alpha)(ku_c'/c)^{8-4\alpha}}{\Gamma(1-\alpha)\Gamma(3-\alpha)|I_0|u_c'^3}, \quad \chi_1 = \chi_0 + \pi(\alpha + \tfrac{1}{2}), \quad \chi_0 = \arg I_0; \tag{3.28a}$$

$$b_2 = \frac{2\pi(1-\alpha)^2 k (ku_c'/c)^{4-4\alpha}}{(1-2\alpha)[\Gamma(1-\alpha)]^2 \Gamma(2-2\alpha)|I_0|u_c'^2}; \tag{3.28b}$$

$$b_3 = \frac{2^{\alpha+3}\pi^{5/2}\alpha^2 k (ku_c'/c)^{8-4\alpha}}{\Gamma(1+\alpha)[\Gamma(1-\alpha)]^5 \Gamma(\frac{1}{2}(2+\alpha))\Gamma(\frac{1}{2}(1-\alpha))|\Delta||I_0|u_c'^4}, \quad \chi_2 = \chi_0 + \arg\Delta - \frac{\pi}{2}. \tag{3.28c}$$

In (3.25)–(3.27), the kernels are real and universal, i.e. they are independent of the flow structure as a whole, while the phases of nonlinear terms, on the contrary, do depend on the flow structure (in terms of χ_0 in (3.25), (3.26) and $\chi_0 + \arg\Delta$ in (3.27)). For the flow models (2.7a, b), the dependence of χ_1 and χ_2 on α for different values of the asymmetry parameter D is shown in figure 9.

The NEEs (3.25) and (3.27) are a generalization of equations (4.1) and (4.2) derived in Paper I, to the case of an arbitrary flow and $\alpha \neq \frac{1}{2}$ (i.e. $R_c \neq \frac{1}{4}$). By contrast, NEE (3.26) is a new one; it is based on the non-dissipative cubic nonlinearity which

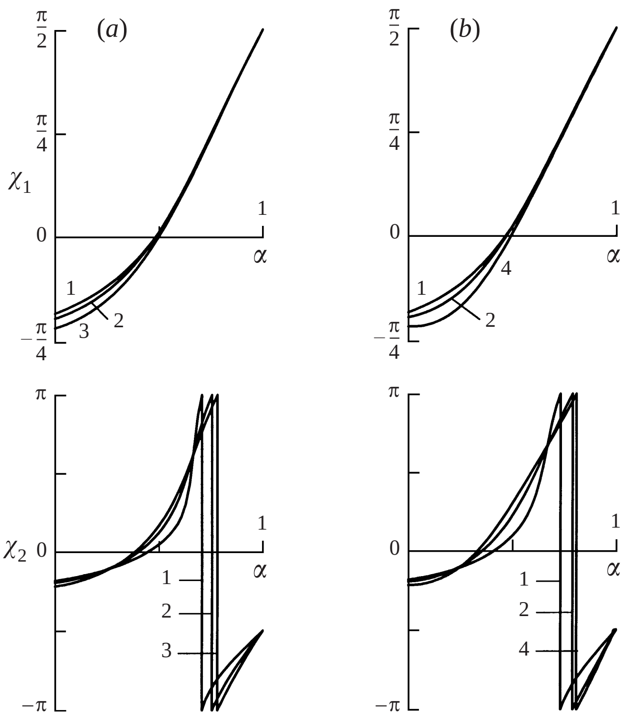

FIGURE 9. Variation of the phases χ_1 and χ_2 along the stability boundary: (*a*) model (2.7*a*), (*b*) model (2.7*b*). Curve 1, $D = 0.1$; 2, $D = 0.3$; 3, $D = 0.5$; 4, $D = 0.49$.

disappears when the parameter $Q = u_c''/u_c' - \alpha r_c'/r_c$ becomes zero (as in the case of Drazin's and Holmboe's flows). In the NEE (3.27) the nonlinear term is written in two forms. Its representation in terms of $H(t, v, w)$ is in the best agreement with the quintic nonlinearity in Paper I whereas the representation in terms of $G(v, w)$ is more suitable for evaluating the parameters of explosive growth (see Appendix D).

Section 4 is devoted to the analysis of the solutions of NEEs (3.25)–(3.27) and of the competition between the nonlinearities. At this point, however, it is pertinent to consider yet another issue associated with the derivation of the NEE.

3.6. *The NEE in the viscous CL regime*

Equations of the inner problem for the viscous CL regime ($\mu \ll \nu^{1/3}$, $\varepsilon \ll \nu^{(2-\alpha)/3}$) can be obtained from (3.2) and (3.3) by simple reordering:

$$y - y_c = \nu^{1/3} Y, \quad \psi = \varepsilon \nu^{\alpha/3} \Psi, \quad \alpha u_c' \rho / r_c = -\varepsilon \nu^{(\alpha-1)/3} P_Y.$$

Retaining only the necessary terms we get

$$\Psi_{YYY} - u_c' Y \Psi_{Yx} + \alpha u_c' \Psi_x = -\varepsilon \nu^{(\alpha-2)/3} (\Psi_{YY} \Psi_x - \Psi_{Yx} \Psi_Y) - (1-\alpha) u_c' \Phi_x, \quad (3.29)$$

$$Pr^{-1} \Phi_{YYY} - u_c' Y \Phi_{Yx} + (1-\alpha) u_c' \Phi_x = -\varepsilon \nu^{(\alpha-2)/3} (\Phi_{YY} \Psi_x - \Phi_{Yx} \Psi_Y) - \frac{Pr-1}{Pr} \Psi_{YYY}.$$

Unlike (3.2) and (3.3), the inner problem (3.29) is not degenerate (except when $Pr = 1$): in any iteration of its solution $\Phi \neq 0$. For this reason only the main sequence should be calculated,

$$\Psi_1 = \left[\Psi_1^{(1)} + \varepsilon^2 \nu^{2(\alpha-2)/3}\Psi_1^{(2)} + \cdots\right]\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

$$\Psi_0 = \varepsilon\nu^{(\alpha-2)/3}\Psi_0^{(1)} + \cdots, \qquad \Psi_2 = \left[\varepsilon\nu^{(\alpha-2)/3}\Psi_2^{(1)} + \cdots\right]\mathrm{e}^{2\mathrm{i}(kx-\omega t)}, \ldots,$$

and the same iterations for Φ. The main nonlinear contribution to the NEE is cubic in A, and it is obtained when matching $\Psi_1^{(2)}$ and $\Phi_1^{(2)}$ to (2.10) and (2.11) respectively. Such a matching to those terms in (2.10) and (2.11) which are $\sim \varepsilon\mu\nu^{(1-\alpha)/3} f^{\pm} Y^{1-\alpha}$ after reordering, calls for the scaling

$$\varepsilon^2 = \mu\nu^{(5-4\alpha)/3}$$

and, because the nonlinearity is obviously local (i.e. the nonlinear term depends only on the current value of the amplitude rather than on the whole history as in (3.25)–(3.27)), gives when substituted into (2.16) a Landau–Stuart–Watson equation of the form

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + \frac{Pr-1}{|I_0|Pr}\frac{b\mathrm{e}^{-\mathrm{i}\chi_3}}{\nu^{(5-4\alpha)/3}}|A|^2 A, \tag{3.30}$$

analogous to those obtained earlier for $\alpha = \frac{1}{2}$ in Drazin's and Holmboe's models ((Brown *et al.* 1981; Churilov & Shukhman 1987); cf. also (1.4)). Here $b = b(\alpha) = O(1)$ is a real constant.

The behaviour of its solutions is well known. As long as the amplitude is small enough, it grows exponentially, and once the nonlinearity threshold

$$|A| = A_1 = O(\gamma_L^{1/2}\nu^{(5-4\alpha)/6}) \tag{3.31}$$

(curve 1 in figures 12*a,b*) is reached, the nonlinearity comes into play and the subsequent course of the evolution is determined by the sign of $[(Pr-1)\,b\cos\chi_3]$. If this sign is negative, the nonlinearity will establish stability at the level (3.31); otherwise, however, the nonlinearity will accelerate the growth of a disturbance and will make it explosive,

$$A \sim (x_0 - x)^{-1/2+\mathrm{i}\beta}, \qquad \beta = \tfrac{1}{2}\tan\chi_3$$

(the lower broad dashed arrow in figure 12*a,b*). The growth rate $\gamma = |A|^{-1}\mathrm{d}|A|/\mathrm{d}x$ increases rapidly with the amplitude, $\gamma = O(|A|^2/\nu^{(5-4\alpha)/3})$, and when

$$A = O(\nu^{1-2\alpha/3}) \tag{3.32}$$

the unsteady scale $l_t = \gamma$ becomes equal to the viscous scale, $l_\nu = \nu^{1/3}$, and the transition to the unsteady CL regime occurs, where the growth is also explosive but with a different exponent (see the next Section and figure 12).

Thus, in the NEE (3.30) two constants, b and χ_3, are unknown, and they can be found by performing an extremely tedious procedure of calculating the necessary iterations of the solution of the inner problem (3.29).† But we may look for another way. First we determine χ_3. As can be seen from (2.16) and (2.17), $\chi_3 = \chi_0 + \chi + \pi/2$ is the sum of contributions from the outer (χ_0) and inner (χ) solutions, and χ is the phase of the jump $(m^+ - m^-)$ of the coefficient at $Y^{1-\alpha}$ in the outer asymptotic

† Churilov & Shukhman (1987) devoted most of their not short a paper to corresponding calculations in the presumably simplest case $\alpha = \frac{1}{2}$.

expansion of Φ_1:

$$m^+ - m^- = -\frac{Pr-1}{Pr}\frac{b_0 \mathrm{e}^{-\mathrm{i}\chi}}{\nu^{(5-4\alpha)/3}}|A|^2 A, \qquad b_0 = \frac{b}{(1-\alpha)k}. \tag{3.33}$$

As in the case $\alpha = \frac{1}{2}$ (see Churilov & Shukhman 1987), in each iteration of the inner solution one can find a certain relation between $\Psi(-Y)$ and $\Psi(Y)$ without solving (3.29). Namely, taking $\Psi_1^{(1)} = AS_1(Y)$, $\Psi_0^{(1)} = |A|^2 S_0(Y)$, $\Psi_2^{(1)} = A^2 S_2(Y)$ and $\Psi_1^{(2)} = |A|^2 AS(Y)$, one can easily obtain that

$$S_1(-Y) = \mathrm{e}^{-\mathrm{i}\pi\alpha}\overline{S_1(Y)}, \quad S_2(-Y) = \mathrm{e}^{-2\mathrm{i}\pi\alpha}\overline{S_2(Y)}, \quad S(-Y) = \mathrm{e}^{-\mathrm{i}\pi\alpha}\overline{S(Y)}$$

and $S_0(Y)$ is an even real function. Using these relations we see that the phase $\chi = \pi\alpha$, i.e. it is the same as in (3.20), and therefore $\chi_3 \equiv \chi_1$. This fact is not surprising because the NEEs (3.25) and (3.30) are the limiting cases (corresponding to $\nu\mu^{-3} \ll 1$ and $\nu\mu^{-3} \gg 1$ respectively) of a more general, viscous-unsteady NEE valid for any relation between μ and ν (evolution equations of this type have been derived for unstratified shear flows, see e.g. Wu, Lee & Cowley 1993; Churilov & Shukhman 1994). In this sense, one can say that (3.25) and (3.30) are continuations of each other in the parameter $\nu\mu^{-3}$.

Now, to determine the character of disturbance evolution in the viscous CL regime we have to know only the sign of b rather than its value. With this in mind, let us consider the NEE (3.25) in the limit of non-competitive nonlinearity when the amplitude is still very small and increases exponentially,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + \frac{Pr-1}{Pr}\nu b_1 \mathrm{e}^{-\mathrm{i}\chi_1} d|A|^2 A, \tag{3.34}$$

$$d = \frac{\Gamma(8-4\alpha)}{(2\gamma_L)^{8-4\alpha}}\int_0^1 \mathrm{d}\sigma \frac{\sigma^{3-2\alpha}G_\nu(\sigma)}{(1+\sigma)^{8-4\alpha}}$$
$$= \frac{1}{(2\gamma_L)^{8-4\alpha}}\left[\frac{2\Gamma(4-2\alpha)}{2-\alpha}(5-12\alpha+5\alpha^2) - \frac{3\Gamma(2-\alpha)\Gamma(\frac{7}{2}-2\alpha)}{2\Gamma(\frac{5}{2}-\alpha)}(3-9\alpha+4\alpha^2)\right],$$

and compare it with (3.30). Being the continuations of each other (in the parameter $\nu\mu^{-3}$), equations (3.30) and (3.34) have the same form, and their nonlinear terms have the same phase and the same dependence on Pr. In (3.34) the factor d becomes zero when $\alpha = \alpha_0 = 0.6886$, is positive when $0 \leqslant \alpha < \alpha_0$ and is negative when $\alpha_0 < \alpha < 1$, and $b_1 > 0$ everywhere in $0 < \alpha < 1$. As for the factor b in (3.30), we know that it is positive when $0 < \alpha \ll 1$ (Shukhman & Churilov 1997) and at $\alpha = \frac{1}{2}$ (Churilov & Shukhman 1987), and it would be reasonable to suggest that $b > 0$ when $0 < \alpha < \alpha_1$, where α_1 is close to α_0 (at least $\alpha_1 > \frac{1}{2}$).

Thus, there is no need to derive the NEE for the viscous CL regime; its functional form (3.30) and the information about χ_3 and $b(\alpha)$ obtained above will suffice for our purposes. We wish to note in conclusion that in the models of Drazin, Holmboe and (2.7a,b) as well as in weakly stratified flows with an arbitrary velocity profile, $|\chi_1| < \pi/2$, i.e. $\cos\chi_1 > 0$, so that the evolution character of not too long-wavelength disturbances is governed by the sign of $(Pr-1)$: stabilization when $Pr < 1$, and destabilization when $Pr > 1$. This regularity was first revealed in symmetric weakly supercritical flows (Brown *et al.* 1981; Churilov & Shukhman 1987), and subsequently it was studied thoroughly by Lott & Teitelbaum (1992).

4. Evolution of unstable disturbances in the unsteady CL regime

4.1. *Particular evolution scenarios*

First we consider what picture of disturbance evolution is given by each of equations (3.25)–(3.27) individually. The right-hand side of each of them involves two terms, linear and nonlinear. The linear term in all equations is the same, and the nonlinear terms closely resemble each other in their structure, which, of course, manifests itself in a similar behaviour of the solutions.

In the full form NEEs (3.25)–(3.27) can be solved numerically only. But in the two limiting cases, linear and nonlinear, one of two terms on the right-hand side of the NEE can be neglected, and then the solution can be found in analytic form. In the linear limit, i.e. far upstream, where the amplitude is still very small (according to (2.3) $A \to 0$ when $x \to -\infty$), this is evidently

$$A(x) = A_0 e^{\gamma_L x} \tag{4.1}$$

(without loss of generality, it will be assumed that $A_0 > 0$). And in the nonlinear limit this is an explosive growth given by the law

$$A = B(x_0 - x)^{-a+i\beta}. \tag{4.2}$$

The general analysis of the problem (Churilov & Shukhman 1992) suggests that in the unsteady CL regime the initial (exponential) growth of unstable disturbances should be accelerated to an explosive one. Such an acceleration has been repeatedly demonstrated by numerical integration of various NEEs of the same type as (3.25)–(3.27) (e.g. Paper I, Goldstein & Choi 1989; Goldstein & Leib 1989; Shukhman 1991), and we have no reason to expect that solutions of (3.25)–(3.27) will behave in a different manner. However, each nonlinearity has its own nonlinearity threshold (i.e. the level of the amplitude, A_{th}, at which the nonlinear term begins to compete with the linear term and the disturbance development begins to depart markedly from (4.1)) and at the nonlinear stage it sets its own rate of evolution (i.e. its own parameter a in (4.2) and its own dependence $\beta(\chi)$). Let us consider these characteristics.

To determine the nonlinearity thresholds we consider the initial stage of development (4.1) and calculate corrections for the nonlinearity by substituting (4.1) into the right-hand side of the corresponding NEE. We obtain:
from (3.25) (cf. (3.34))

$$\frac{dA}{dx} = \gamma_L A + \frac{Pr-1}{Pr}\frac{\nu d_1 e^{-i\chi_1}}{\gamma^{8-4\alpha}}|A|^2 A, \qquad d_1 = O(1); \tag{4.3a}$$

from (3.26)

$$\frac{dA}{dx} = \gamma_L A + \frac{Q d_2 e^{-i(\chi_1+\pi/2)}}{\gamma^{4-4\alpha}}|A|^2 A, \qquad 0 < d_2 = O(1); \tag{4.3b}$$

from (3.27)

$$\frac{dA}{dx} = \gamma_L A + \frac{d_3 e^{-i\chi_2}}{\gamma^{8-4\alpha}}|A|^4 A, \qquad 0 < d_3 = O(1); \tag{4.3c}$$

where $\gamma = \gamma_L$. By order-of-magnitude equating the linear and the nonlinear terms in (4.3*a–c*), we find the nonlinearity threshold for each NEE:
for the dissipative nonlinearity (NEE (3.25)) – an extension of (3.31) to the region $\gamma_L > \nu^{1/3}$,

$$|A_{th}| = A_2 = O(\gamma_L^{9/2-2\alpha}\nu^{-1/2}); \tag{4.4a}$$

for the non-dissipative cubic nonlinearity (NEE (3.26))

$$|A_{th}| = A_3 = O(\gamma_L^{5/2-2\alpha}); \tag{4.4b}$$

for the non-dissipative quintic nonlinearity (NEE (3.27))

$$|A_{th}| = A_4 = O(\gamma_L^{9/4-\alpha}). \tag{4.4c}$$

It is interesting to note that equations (4.3*a–c*) also describe qualitatively correctly the explosive stage if γ means a current growth rate, $\gamma = |A|^{-1}\mathrm{d}|A|/\mathrm{d}x$. In particular, they allow one to determine the rates of evolution at the nonlinear stage:
for the dissipative nonlinearity (NEE (3.25))

$$\gamma^{(\nu)} = O\left[(\nu|A|^2)^{1/(9-4\alpha)}\right], \quad \text{or} \quad a = \tfrac{9}{2} - 2\alpha; \tag{4.5a}$$

for the non-dissipative cubic nonlinearity (NEE (3.26))

$$\gamma^{(Q)} = O\left(|A|^{2/(5-4\alpha)}\right), \quad \text{or} \quad a = \tfrac{5}{2} - 2\alpha; \tag{4.5b}$$

for the non-dissipative quintic nonlinearity (NEE (3.27))

$$\gamma^{(5)} = O\left(|A|^{4/(9-4\alpha)}\right), \quad \text{or} \quad a = \tfrac{9}{4} - \alpha. \tag{4.5c}$$

Finally, we substitute (4.2) into an appropriate NEE to ascertain that it is indeed the solution (when $\gamma_L = 0$) and the exponent a is correctly calculated by (4.3); furthermore, we obtain equations for determining $\beta(\chi)$ (see (D 5), (D 6) and (D 12)). Results derived from solving them are presented in figure 10. While referring the reader to Appendix D for details, at this point we emphasize that $\beta(\chi)$ is generally many-valued (this property was originally pointed out by Shukhman 1991), on the interval $-\pi < \chi \leqslant \pi$ it becomes zero at $\chi = \chi_*$, and is an odd 2π-periodic function of $(\chi - \chi_*)$. The parameter χ_* can take one of two values: either $\chi_* = 0$ or $\chi_* = \pi$. For non-dissipative nonlinearities (NEEs (3.26) and (3.27)), we have $\chi_* = 0$, and for the dissipative nonlinearirity we have $\chi_* = 0$ when $0 \leqslant \alpha < \alpha_*$ and $\chi_* = \pi$ when $\alpha_* < \alpha < 1$. Such a jump of χ_* is due to the fact that when $\alpha = \alpha_* = 0.615$ the integral in (D 2) changes sign.

The solutions of NEEs (3.25)–(3.27) behave in the same manner qualitatively; therefore, we give a general description and illustrate it with the results of a numerical integration of NEE (3.26) with the 'initial condition' (4.1) at different values of the phase χ of the nonlinear term (figure 11). NEE (3.26) is chosen because its kernel is the simplest for numerical calculation. After transforming (3.26) to the form (D 3) (with $G(\sigma) \equiv G_Q(\sigma)$, $\lambda = 4 - 4\alpha$ and $\mu = 3 - 2\alpha$) which is more convenient for integrating, all the information on the outer solution (i.e. on the flow structure) is contained only in the phase χ. For this reason, it is appropriate to investigate the dependence of the behaviour of the NEE solution on χ rather than to solve the NEE for a particular value of χ obtained in some model of the flow (such as (2.7)).

In the general case the evolution path on the plane of a complex A represents an unwinding spiral (see figure 11*c, d*), in full accord with (4.2). The asymptotic growth rate of the amplitude is determined by the parameter a (see (4.5)), and the direction and tightness of the spiral winding is governed by the parameter β. In view of the multivaluedness of $\beta(\chi)$, only a numerical calculation can show which of its branches will be reached in the course of evolution of the solution starting from (4.1). In all our calculated variants, the solution for $|\chi - \chi_*| < \pi$ reaches the main branch on which $\beta = 0$ when $\chi = \chi_*$.

Special attention should be given to the cases $\chi = 0$ and $\chi = \pm\pi$ when the

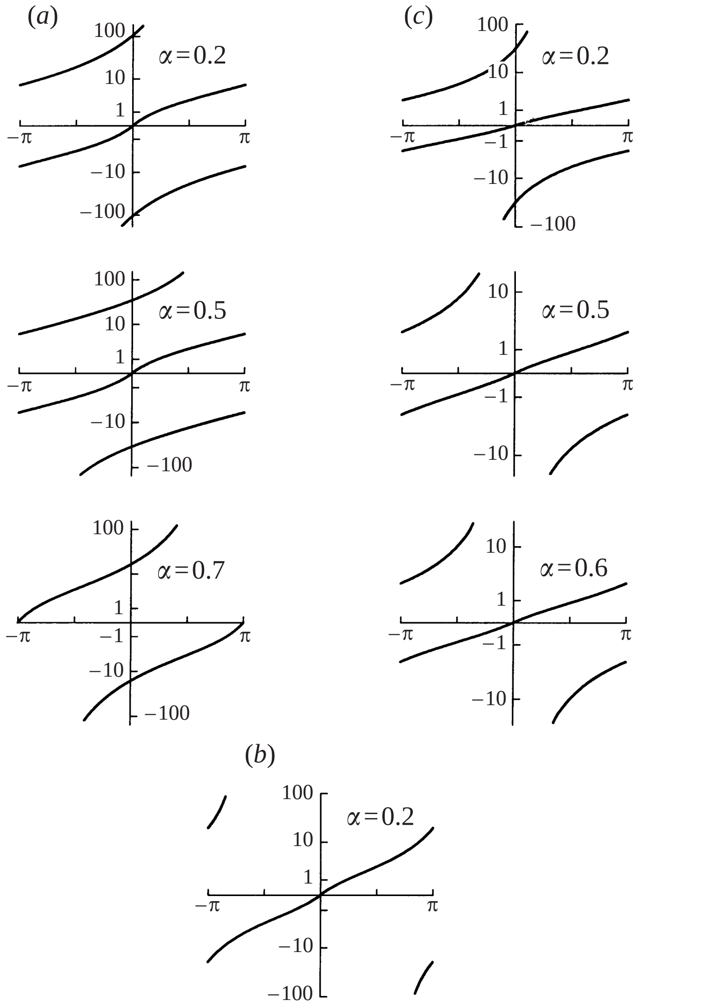

FIGURE 10. Dependence $\beta(\chi)$ for NEEs (*a*) (3.25); (*b*) (3.26); (*c*) (3.27).

NEE is real and must therefore have real solutions. The case $\chi = \chi_*$ (to which the 'destabilizing' nonlinearity in (4.3) corresponds) fits well into the general picture described above: when $\beta = 0$ the spiral (4.2) 'straightens', and the amplitude A remains real and grows monotonically in an explosive manner (figure 11*a*).

When $\chi - \chi_* = \pm\pi$ the values of β are represented by pairs, $\pm\beta_i$, and they include no zero values (see figure 10), so all asymptotic representations of the form (4.2) are necessarily complex. The solution of the NEE in this case, however, is real and, as shown in Paper I, represents oscillations of A around zero whose frequency and amplitude increase explosively (figure 11*b*). Asymptotically, this looks like a superposition of two solutions (4.2), with $+\beta$ and $-\beta$, although in the nonlinear equation the superposition of solutions is obviously not a solution. It is interesting to note that when $|\chi - \chi_*| \to \pi$, as a numerical calculation shows, A performs a series of oscillations and only after that it does reach the spiral (4.2) (figure 11*e*; see also figure 5*b* in Shukhman 1991).

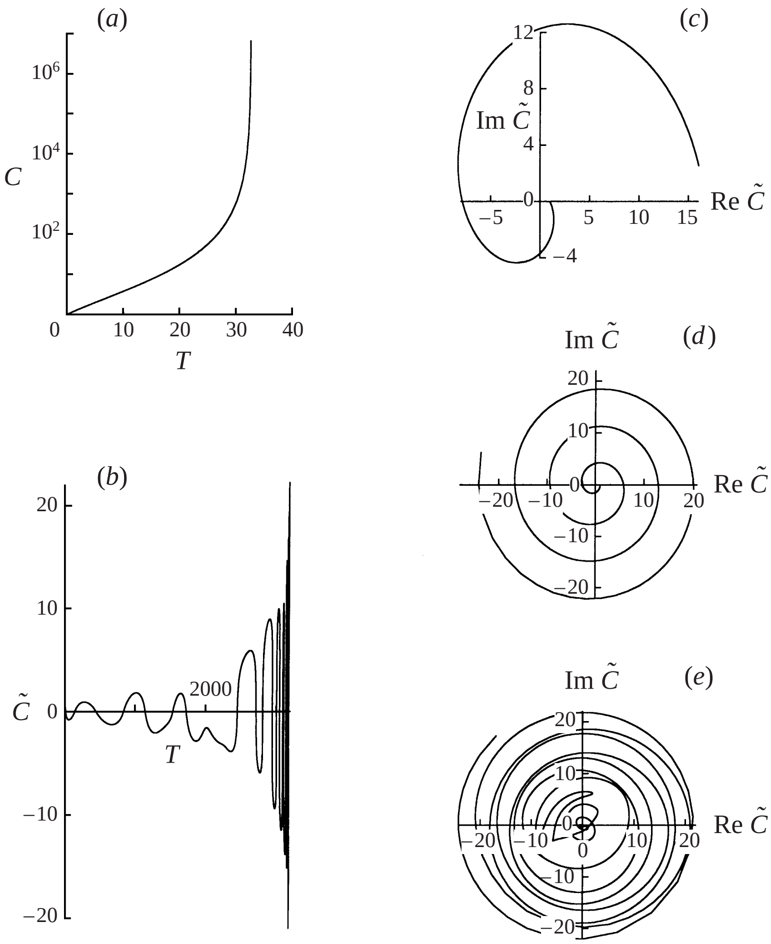

FIGURE 11. Results of an integration of NEE (3.26) $(\chi = \chi_1 + \pi/2,\ \ \alpha = 0.2)$: (*a*) $\chi = 0$, (*b*) $\chi = \pi$, (*c*) $\chi = \pi/4$, (*d*) $\chi = \pi/2$, (*e*) $\chi = 0.9\pi$; C, $\tilde{C}$ and T are defined in Appendix D.

4.2. *Competition between the nonlinearities and a general evolution scenario*

Now that we have considered each individual nonlinearity, it is appropriate to abandon their artificial separation and write a single nonlinear evolution equation for the unsteady CL regime

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} &= \gamma_L A + \frac{Pr-1}{Pr} v b_1 \mathrm{e}^{-\mathrm{i}\chi_1} \int_0^\infty \mathrm{d}s\, s^{7-4\alpha} \int_0^1 \mathrm{d}\sigma \sigma^{3-2\alpha} G_v(\sigma) A(x-s) A(x-\sigma s) \\ &\times \overline{A}(x-(1+\sigma)s) + Q b_2 \mathrm{e}^{-\mathrm{i}(\chi_1+\pi/2)} \int_0^\infty \mathrm{d}s\, s^{3-4\alpha} \int_0^1 \mathrm{d}\sigma\, \sigma^{2-2\alpha} G_Q(\sigma) A(x-s) \\ &\times A(x-\sigma s) \overline{A}(x-(1+\sigma)s) + b_3 \mathrm{e}^{-\mathrm{i}\chi_2} \int_0^\infty \mathrm{d}s\, s^{7-4\alpha} \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}v \int_0^1 \mathrm{d}w\, H(t,v,w) \\ &\times A(x-s) A(x-ts) A(x-tvs) \overline{A}(x-2tvws) \overline{A}(x-(1+t+tv-2tvw)s), \qquad (4.6)\end{aligned}$$

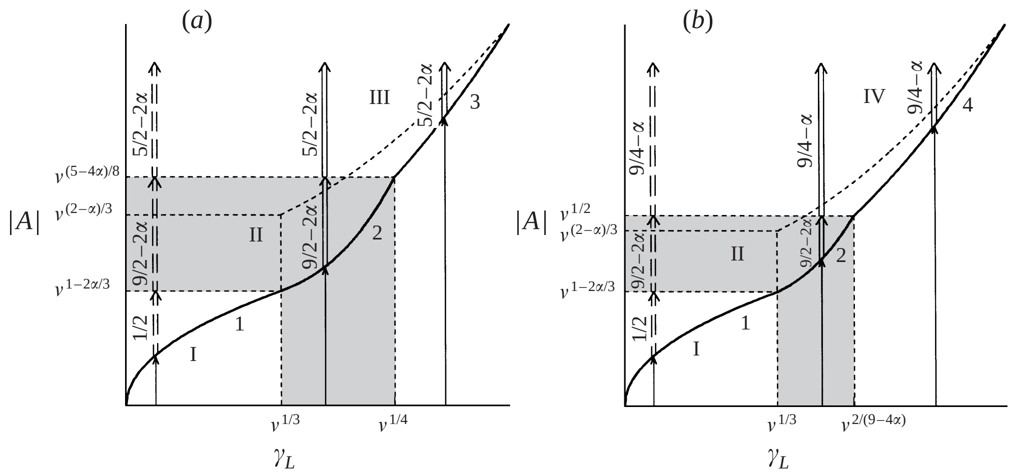

FIGURE 12. Evolution scenarios for disturbances on the $\gamma_L - A$ diagram: (*a*) $0 < \alpha < \frac{1}{4}$, (*b*) $\frac{1}{4} < \alpha < \frac{3}{4}$. Viscous CL region: I. Unsteady CL region: II (shaded), dissipative nonlinearity is dominant; III, nondissipative cubic nonlinearity is dominant; IV, quintic nonlinearity is dominant. Nonlinearity threshold: Curve 1, $A = A_1$, 2, $A = A_2$, 3, $A = A_3$, 4, $A = A_4$. Arrows: $\longrightarrow$ $A \sim \exp(\gamma_L x)$; $\overset{a}{\Longrightarrow}$ $|A| \sim (x_0 - x)^{-a}$.

and to investigate the behaviour of its solutions for different levels of supercriticality γ_L (i.e. for different pumping frequencies ω).

Far upstream, nonlinear terms in (4.6) are non-competitive, and the amplitude grows as (4.1). The nonlinearity with the lowest threshold (at a given γ_L), will obviously be the first to intervene in the instability development, and it will also determine the disturbance behaviour in the initial period of the nonlinear stage of evolution. Subsequently, however, the winner in the competition could be another nonlinearity, that which at a given amplitude level ensures the fastest growth rate, i.e. the largest γ. The other two nonlinear terms, however, will give only small corrections to the growth rate dictated by this dominant nonlinearity.

The comparison of the non-dissipative nonlinearities

$$A_3/A_4 = O(\gamma_L^{1/4-\alpha}), \qquad \gamma^{(Q)}/\gamma^{(5)} = O\left(|A|^{2(4\alpha-1)/[(5-4\alpha)(9-4\alpha)]}\right), \tag{4.7}$$

shows that, irrespective of the amplitude level, the cubic nonlinearity is stronger than the quintic one when $\alpha < \frac{1}{4}$, and when $\alpha > \frac{1}{4}$ the quintic nonlinearity is stronger. Upon comparing (4.4*b, c*) with (4.4*a*), we find the nonlinearity threshold for the whole equation (4.6):
for $\alpha < \frac{1}{4}$ (figure 12*a*)

$$|A_{th}| = \begin{cases} A_2 = O(\gamma_L^{9/2-2\alpha}\nu^{-1/2}), & \nu^{1/3} < \gamma_L < \nu^{1/4}, \\ A_3 = O(\gamma_L^{5/2-2\alpha}), & \nu^{1/4} < \gamma_L < 1; \end{cases} \tag{4.8a}$$

for $\frac{1}{4} < \alpha < \frac{3}{4}$ (figure 12*b*)

$$|A_{th}| = \begin{cases} A_2 = O(\gamma_L^{9/2-2\alpha}\nu^{-1/2}), & \nu^{1/3} < \gamma_L < \nu^{2/(9-4\alpha)}, \\ A_4 = O(\gamma_L^{9/4-\alpha}), & \nu^{2/(9-4\alpha)} < \gamma_L < 1. \end{cases} \tag{4.8b}$$

Recall that when $\gamma_L < \nu^{1/3}$ the nonlinearity threshold is defined by the relationship (3.31) (curve 1 in figure 12*a, b*).

On comparing the evolution rates (4.5*b, c*) with (4.5*a*), it will readily be seen that wherever the dissipative nonlinearity threshold is lower and this nonlinearity is the first to come into play, the non-dissipative nonlinearity during the course of the evolution would of necessity become stronger than it. This occurs at a level

$$A = O(\nu^{(5-4\alpha)/8}) \quad \text{if} \quad \alpha < \tfrac{1}{4} \tag{4.9a}$$

or

$$A = O(\nu^{1/2}) \qquad \text{if} \quad \tfrac{1}{4} \leqslant \alpha < \tfrac{3}{4}. \tag{4.9b}$$

This rule also extends to the unsteady CL region which appears when $\gamma_L < \nu^{1/3}$ (above the viscous CL region, see figure 12) in the case when the nonlinearity in NEE (3.30) plays a destabilizing role and, once the instability threshold (3.31) is reached, it accelerates the growth of disturbances to an explosive one. The transition from the viscous CL to the unsteady CL occurs at a level (3.32), and immediately after the transition the dissipative nonlinearity is dominant; subsequently, however, one of the non-dissipative nonlinearities will necessarily become the main one: cubic (at a level (4.9*a*)) if $\alpha < \frac{1}{4}$ or quintic (at a level (4.9*b*)) if $\frac{1}{4} < \alpha < \frac{3}{4}$.

The scenarios outlined above, with all the changes of the dominant nonlinearity and, accordingly, of the evolution law are represented on the γ_L, A diagrams (figure 12*a, b*). For this purpose, the unsteady CL region is divided into subregions (II and III in figure 12*a*, and II and IV in figure 12*b*) according to where each particular nonlinearity is dominant, and the index a of the explosive growth rate (4.2) is shown near the arrows. Notice that with increasing α, the size of subregion II where the dissipative nonlinearity is dominant, decreases.

Finally, we now discuss the validity range of NEE (4.6). It is derived by means of perturbation theory which is valid so long as the inequalities are satisfied (see (3.5) and (3.16)):

$$F_0 = \varepsilon^2\mu^{-3} = O(|A|^2/\gamma^3) \ll 1 \text{ and } F_1 = \varepsilon^2\mu^{2(\alpha-2)} = O(|A|^2\gamma^{2(\alpha-2)}) = O[(l_N/l_t)^{2(2-\alpha)}] \ll 1.$$

Let us consider first the case when stratification is not weak. The unsteady scale $l_t = \gamma$ is determined by the largest of the growth rates (4.5*a–c*) and therefore $l_t \geqslant \gamma^{(5)}$ in any case. On the other hand, $\gamma^{(5)} \gg l_N$ when $|A| \ll 1$ and hence $l_t \gg l_N$ right up to $A = O(1)$, i.e. to the validity boundary of weakly nonlinear theory. In other words, if α is not too small the quintic nonlinearity does guarantee the fulfilment of the inequality $F_1 \ll 1$ which means that, as in Paper I, the nonlinear CL regime does not arise evolutionarily.

As far as the parameter F_0 is concerned, however, it is obviously small in those regions of the γ_L, A diagram where cubic nonlinearities are dominant, but in region IV (figure 12*b*) where the evolution behaviour is determined by the quintic nonlinearity and $\gamma = \gamma^{(5)}$, F_0 is no longer small if $\alpha \geqslant \frac{3}{4}$. Breakdown of the inequality $F_0 \ll 1$ means that the contribution to the NEE from the main sequence (see § 3.2) becomes 'strongly nonlinear' (it must be emphasized that the CL regime remains unsteady in this case, i.e. $l_t \gg l_N$): it will be a sum of the same order contributions from an infinite number of iterations, and the perturbation theory developed here is unsuitable for calculating it. Thus, NEE (4.6) is valid only if $\alpha < \frac{3}{4}$. The value of $\alpha = \frac{3}{4}$ is also distinguished by the fact that it is at this α that the levels (3.32) and (4.9*b*) merge together and the dissipative nonlinearity becomes non-competitive everywhere (subregion II in figure 12*b* disappears).

On the other hand, in the case of a weak stratification ($\alpha \ll 1$) only the inequality $F_1 \ll 1$ can be violated because $F_0 \ll F_1$. This violation becomes possible as all three nonlinearities in (4.6) are weakened when α decreases, and it makes transition to the nonlinear CL regime in a weakly stratified flow inevitable. The case of weak stratification was studied in detail by Shukhman & Churilov (1997). And here it is worth to note that in NEE (4.6) $b_1 = O(\alpha)$, $Q = O(\alpha)$ because, as can be easily shown, $u_c''/u_c' = O(\alpha)$ when $\alpha \to 0$ and $b_3 = O(\alpha^2)$ so that the quintic nonlinearity reduces much faster than cubic nonlinearities and cannot compete with them.

5. Discussion

Thus, in terms of a weakly nonlinear theory we have studied the spatial evolution of weakly unstable monochromatic disturbances (created by an external source far upstream) in stably stratified shear flows of the mixing layer type with sufficiently arbitrary velocity and density profiles. We now summarize and discuss the main results.

1. NEE (4.6) has been derived which describes the development of disturbances in the unsteady CL regime. It is a generalization of NEE (5.1) (henceforth referred to as (I.5.1)) obtained in Paper I to the case of asymmetric flows and an arbitrary level of stratification (i.e. an arbitrary value of the Richardson number in the range $0 < R_c < \frac{1}{4}$).

The study made in Paper I involved two main restrictions which have now been successfully removed: the flow model was assumed to be symmetric (or, more specifically, all calculations were performed for Drazin's model (1.1)), and $R_c \approx \frac{1}{4}$ ($\alpha \approx \frac{1}{2}$), which in symmetric flows corresponds to the top of a neutral curve. In this paper, we, in the first place, have considered arbitrary α ($0 < \alpha < \frac{3}{4}$), and we have included nearly the entire stability boundary, except for its longest-wavelength part ($\frac{3}{4} < \alpha < 1$). Secondly, relaxing the symmetry assumption, we have detected a new, non-dissipative cubic nonlinearity which, along with the other two known from Paper I (dissipative and non-dissipative quintic) governs the instability development. The analysis has shown that the decisive role in the disturbance evolution is played by this nonlinearity in the case of an intermediate stratification when $\alpha \leqslant \frac{1}{4}$ ($R_c \leqslant \frac{3}{16}$), while in the case of a stronger stratification ($\frac{3}{16} < R_c < \frac{1}{4}$ or $\frac{1}{4} < \alpha < \frac{3}{4}$) it cannot compete with the quintic nonlinearity and gives only small corrections to the evolution law.

As a result, it turns out that the validity range of NEE (I.5.1) is broader than would be suspected: the limitation on the stratification ($R_c \approx \frac{1}{4}$, $\alpha \approx \frac{1}{2}$) remains the same but the flow symmetry is not very important. For (I.5.1) to be valid also for asymmetric flows, it is necessary merely to restore the phase factors $e^{-i\chi_1}$ and $e^{-i\chi_2}$ (determined entirely by the outer problem, see (3.28)) in nonlinear terms thus making them complex, as discussed in Paper I. It must be emphasized, however, that in the general case $R_c \approx 1/4$ does not imply weak supercriticality of the flow (i.e. a nearly total suppression of the instability by stratification, see figure 4), and in weakly supercritical flows R_c, in turn, can differ considerably from $\frac{1}{4}$. In this and all other cases of a substantial difference of R_c from $\frac{1}{4}$, NEE (I.5.1) is unsuitable, even if its modifications are taken into account, and NEE (4.6) should be used.

Because of the 'universality' of the nonlinear terms in (4.6), applying it to a particular flow requires little effort. To determine the values of all parameters involved in (4.6), it is necessary (i) to solve a simple linear problem (2.4) and determine the parameters

(c, y_c, α, and others) and the eigenfunction $g(y)$ of the neutral mode, (ii) to calculate – using them – the integrals I_1, I_2 and I_3 appearing in (2.16), and (iii) to solve the problem (2.4) for the second harmonic (to the left and right of y_c) and calculate the parameter Δ (see (3.15)).

2. NEE (3.30) has been obtained (up to the coefficient $b = O(1)$) to describe the instability development in the viscous CL regime, and it was determined under which conditions the nonlinearity stops the growth of disturbances at the nonlinearity threshold (3.31) and under which it accelerates this growth to an explosive one.

Without resorting to a (very laborious) formal deriving of the NEE, we have succeeded in obtaining all the information needed for determinating the nonlinearity threshold as well as for a (detailed enough) description of disturbance evolution. In particular, it has been shown that in the case of an arbitrarily stratified mixing layer and lack of symmetry the boundary between stabilization and destabilization is at the same value of the Prandtl number, $Pr = 1$, as in the case of symmetric or weakly stratified flows studied earlier.

3. For the unsteady CL regime, a qualitative study was made of the development of unstable disturbances under the control of each of the three nonlinearities individually (NEEs (3.25)–(3.27)) and this study was illustrated by results of numerical calculations.

In their structure, NEEs (3.25)–(3.27) are typical for the unsteady CL regime, as also is the course of the evolution described by them: from the stage of exponential growth (4.1) to an explosive growth 'on a spiral path' (4.2) through some transition stage. The evolution laws (4.1) and (4.2) are exact solutions, respectively, in the linear and nonlinear limits when only one (linear or nonlinear) term is left over on the right-hand side of NEE, and they describe sufficiently clearly the initial and final stages of disturbance development. But to gain a more accurate picture of the evolution as a whole, they must be 'matched', and one cannot manage in this case without a numerical integration of the corresponding NEE. Only in this way it is possible to relate the 'initial' amplitude A_0 and the 'final' coordinate x_0 (in (4.1) and (4.2) they are arbitrary because of the translational invariance of NEE), choose out of several possible values of the parameter of the spiral winding β the value which corresponds to (4.1), and to study in detail the transition stage which can be very complicated and long-lasting (and in the limiting case $|\chi| = \pi$ the solution does not reach (4.2) at all, and the 'transition stage' lasts up to the singularity).

4. The competition between the nonlinearities in the process of instability development was studied and evolution scenarios constructed.

The evolution of disturbances is fast, as it should be in the case of a singular neutral mode (Churilov & Shukhman 1992): the growth is explosive up to $A = O(1)$, the nonlinear CL regime does not arise (except in the case of a weak stratification $\alpha \ll 1$), and only with a very small supercriticality ($\gamma_L \ll \nu^{1/3}$) and when $Pr < 1$ is a stabilization possible in the viscous CL regime. The distinctive features of the problem under consideration appear in the nature of the nonlinearities and in the fact that in the case of a sufficiently large amplitude the dominant nonlinearity is of necessity non-dissipative.

5. The validity range of NEE (4.6) has been determined.

It turns out that along with the traditional (for weakly nonlinear theory) limitations ($|A| \ll 1$, $\gamma_L \ll 1$), the condition $\alpha < \frac{3}{4}$ must be satisfied, i.e. the disturbance must be not too a long-wavelength one. Otherwise, a calculation of the nonlinearity would require a quite different technique enabling us to sum infinite series of perturbation theory.

I am indebted to Dr I. G. Shukhman for helpful discussions. Thanks are also due to Mr V. G. Mikhalkovsky for his assistance in preparing the English version of the manuscript.

The research described in this publication was done under the grant of the Russian Foundation for Fundamental Research No. 96-05-64429.

Appendix A. Solution of the equation $\hat{\mathrm{N}}F = R$

The operators $\hat{\mathrm{N}}^{(a)}$ and $\hat{\mathrm{N}}^{(b)}$ on the left-hand sides of (3.2) and (3.3) differ by the last term only; therefore, we consider the equation

$$\hat{\mathrm{N}}F \equiv \left\{ \left[\frac{\partial}{\partial \tau} + (u_c' Y - c_1)\frac{\partial}{\partial x} \right] \frac{\partial}{\partial Y} - \beta u_c' \frac{\partial}{\partial x} \right\} F = R, \quad 0 < \beta < 1,$$

which is their generalization. For the lth harmonic ($R, F \sim \mathrm{e}^{\mathrm{i}lkx}$)

$$\hat{\mathrm{N}} \equiv \hat{\mathrm{N}}_l = \left[\frac{\partial}{\partial \tau} + \mathrm{i}lk(u_c' Y - c_1) \right] \frac{\partial}{\partial Y} - \mathrm{i}\beta lku_c'.$$

If $l = 0$, the solution of the equation $\hat{\mathrm{N}}_l F = R$ is determined by a direct integration, and all cases $l \neq 0$ are brought to $l = 1$ by substitution $lk = q$.

A.1. *Homogeneous equation* $\hat{\mathrm{N}}_1 G = 0$

On performing a Fourier-transform in τ:

$$G(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}, \qquad g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau G(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau},$$

we obtain

$$z g_Y - \beta g = 0 \quad \text{and} \quad g = a(\omega) z^{\beta},$$

where $z = Y - c_1/u_c' + \omega/(ku_c')$, and $A(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, a(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}$ is an arbitrary function.

It is convenient to start a calculation of G with a calculation of G_Y:

$$G_Y = \beta \int_{-\infty}^{\infty} \mathrm{d}\omega\, a(\omega) z^{\beta-1}\mathrm{e}^{\mathrm{i}\omega\tau} = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\, A(t) \int_{-\infty}^{\infty} \mathrm{d}\omega\, z^{\beta-1}\mathrm{e}^{\mathrm{i}\omega(\tau-t)}.$$

Since all disturbances are advected by the flow, the value of G_Y at a given point $\xi = c\tau$ is determined by events upstream (when $t < \tau$) and are independent of events downstream ($t > \tau$). Passing on to a consideration of complex values of ω we see that this condition requires analyticity of the integrand in the lower half-plane ($\mathrm{Im}\,\omega < 0$). A little rearrangement yields

$$G_Y = \frac{\beta \exp(\mathrm{i}\pi\beta/2)}{2\pi(ku_c')^{\beta-1}} \left(1 - \mathrm{e}^{-2\mathrm{i}\pi\beta}\right) \Gamma(\beta) \int_0^{\infty} \mathrm{d}t\, t^{-\beta} A(\tau - t)\mathrm{e}^{-\mathrm{i}k(u_c' Y - c_1)t}$$

$$= \frac{\Gamma(\beta+1)\exp(\mathrm{i}\pi\beta/2)}{2\pi(ku_c')^{\beta-1}} \int_C \mathrm{d}t\, t^{-\beta} A(\tau - t)\mathrm{e}^{-\mathrm{i}k(u_c' Y - c_1)t},$$

where $\Gamma(z)$ is the Euler's gamma-function (Abramowitz & Stegun 1964), and the contour C is shown in figure 7(a). Upon integrating over Y we obtain

$$G(Y,\tau) = \frac{\Gamma(\beta+1)\exp[\mathrm{i}\pi(\beta+1)/2]}{2\pi(ku_c')^{\beta}} \int_C \mathrm{d}t\, t^{-\beta-1} A(\tau - t)\mathrm{e}^{-\mathrm{i}k(u_c' Y - c_1)t}. \tag{A 1}$$

In the lower half-plane ($\mathrm{Im}\, Y \leqslant 0$) G is analytic, has the asymptotic representation as $|Y| \to \infty$

$$G = A(\tau)Y^{\beta} - \frac{\beta}{u_c'}\left(c_1 + \frac{\mathrm{i}}{k}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)AY^{\beta-1} + O(Y^{\beta-2}) \tag{A 2}$$

and tends to zero when $\tau \to -\infty$ (if $A(\tau) \to 0$). In the upper half-plane $G(Y)$ has singularities.

A.2. *Equation* $\hat{\mathrm{N}}_1 F = R$

In the ω-representation

$$zf_Y - \beta f = -\frac{\mathrm{i}r}{ku_c'}, \qquad f = -\frac{\mathrm{i}z^{\beta}}{ku_c'}\int_{-\infty}^{Y}\frac{\mathrm{d}s\, r(s,\omega)}{z^{\beta+1}}. \tag{A 3}$$

If we pass on to the τ-representation, a simple calculation gives

$$\begin{aligned} F &= \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\beta}}{2\pi}\int_0^{\infty}\mathrm{d}t\int_0^{\infty}\mathrm{d}s\, R(Y-s,\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t}\int_L \mathrm{d}v\, v^{\beta}(1-v)^{-\beta-1}\mathrm{e}^{\mathrm{i}ku_c'stv} \\ &= \int_0^{\infty}\mathrm{d}t\int_0^{\infty}\mathrm{d}s\, R(Y-s,\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t}\Phi(1+\beta,1;\mathrm{i}ku_c'st) \end{aligned} \tag{A 4}$$

where $\Phi(a,c;z)$ is a Kummer's confluent hypergeometric function (Erdelyi 1953), and the contour L is shown in figure 7(b).

This is a particular solution of the non-homogeneous equation. To obtain a general solution, it is necessary to add an arbitrary solution (A 1) of the homogeneous equation.

A.3. *Asymptotic representation*

If $R \to 0$ when $Y \to \pm\infty$, then the asymptotic representation of the particular solution F when $Y \to -\infty$ is determined by the behaviour of R as $Y \to -\infty$. When $Y \to +\infty$, from (A 3) we have

$$f = -\frac{\mathrm{i}z^{\beta}}{ku_c'}\int_{-\infty}^{\infty}\frac{\mathrm{d}Y\, r(Y,\omega)}{z^{\beta+1}} + O(Y^{\beta-1}),$$

and upon passing on to the τ-representation, we obtain

$$F = \frac{(ku_c'Y)^{\beta}\mathrm{e}^{\mathrm{i}\pi\beta/2}}{\Gamma(1+\beta)}\int_{-\infty}^{\infty}\mathrm{d}Y\int_0^{\infty}\mathrm{d}t\, t^{\beta}R(Y,\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t} + O(Y^{\beta-1}). \tag{A 5}$$

The general solution, however, has the obvious asymptotic representation

$$\begin{aligned} F &= C^{\pm}(\tau)Y^{\beta} + O(Y^{\beta-1}), \\ C^{+}(\tau) - C^{-}(\tau) &= \frac{(ku_c')^{\beta}\mathrm{e}^{\mathrm{i}\pi\beta/2}}{\Gamma(1+\beta)}\int_{-\infty}^{\infty}\mathrm{d}Y\int_0^{\infty}\mathrm{d}t\, t^{\beta}R(Y,\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t}. \end{aligned} \tag{A 6}$$

Note that if $R \to 0$ as $|Y| \to \infty$ and is analytic in the lower half-plane ($\mathrm{Im}\, Y < 0$), then $C^{+} = C^{-}$.

If R is not localized (i.e. does not tend to zero when $Y \to \pm\infty$), a calculation of the asymptotic expansions calls for a special consideration. By way of example let us consider a problem with important implications for this study:

$$\hat{\mathrm{N}}_1 F = \mathrm{i}ku_c'G_{\mu}, \tag{A 7}$$

where G_{μ} is obtained from (A 1) by substituting μ for β. If $\mu \neq \beta$, then, it is easy to see that

$$\hat{\mathrm{N}}_1 G_\mu = -\mathrm{i}ku_c'(\beta-\mu)G_\mu, \quad \text{and} \quad F = \frac{G_\mu}{\mu-\beta}. \tag{A 8}$$

The asymptotic expansion of F immediately follows from (A 2). If $\mu = \beta$, it can be readily seen that

$$F = -\frac{\Gamma(\beta+1)\exp[\mathrm{i}\pi(\beta+1)/2]}{2\pi(ku_c')^\beta}\int_C \mathrm{d}t\, t^{-\beta-1}\ln t A(\tau-t)\mathrm{e}^{-\mathrm{i}k(u_c'Y-c_1)t}. \tag{A 9}$$

We wish to point out in conclusion that results for an arbitrary $l \neq 0$ are obtained by substituting of lk for k.

Appendix B. Kernels of the evolution equations (3.25)–(3.27)

This Appendix is available from the *Journal of Fluid Mechanics* Editorial Office.

Appendix C. Some details concerning q^+ and q^-

To obtain a general solution of the non-homogeneous linear equation, it is necessary to add to its particular solution a general solution of the homogeneous equation. The last term in (2.12) represents the inner ($y \to y_c$) expansion of the solutions $g_2^\pm$ of the (homogeneous) Taylor–Goldstein equation for the second harmonic,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{J_N r(y)}{(u-c)^2} - \frac{u''}{u-c} - \frac{(2\omega_N)^2}{c^2}\right] g_2^\pm = 0. \tag{C 1}$$

The function g_2^+ (g_2^-) is defined for $y > y_c$ ($y < y_c$) and is bounded when $y \to +\infty$ ($y \to -\infty$). As $y \to y_c$, the functions $g_2^\pm$ have the expansions

$$g_2^\pm = d_1^\pm |s|^\alpha + d_2^\pm |s|^{1-\alpha} + \cdots, \qquad s = y - y_c \tag{C 2}$$

with the real coefficients $d_1^\pm$ and $d_2^\pm$. Since the non-trivial solution of (C 1) exists both to the right and left of y_c, in each of the pairs (d_1^+, d_2^+) and (d_1^-, d_2^-) at least one of the numbers is non-zero. In the general case all four numbers are non-zero; then a comparison of (C 2) with the last term in (2.12) shows that (we assume that $\arg s = -\pi$ when $y < y_c$)

$$q^+ = d_1^+/d_2^+, \qquad q^- = -(d_1^-/d_2^-)\mathrm{e}^{2\mathrm{i}\pi\alpha}. \tag{C 3}$$

Consider the cases where singularities arise in (3.15) and/or (C 3).

(*a*) If q^+ and q^- are finite and are both non-zero, the singularity in (3.15) appears when $q^+ = q^-$. This equality is possible (and, as will be shown in what follows, necessary) if only $\alpha = \frac{1}{2}$. When $\alpha = \frac{1}{2}$, however, $g_2^\pm$ have the inner expansions (see (2.7) in Paper I)

$$g_2^\pm = |s|^{1/2}(\tilde{q}^\pm + \ln|s|/2) + \cdots .$$

For it to be the limit of (C 2) when $\alpha \to \frac{1}{2}$, it is necessary that

$$d_1^\pm = -(1-2\alpha)^{-1} + \tfrac{1}{2}\tilde{q}^\pm + O(1-2\alpha), \qquad d_2^\pm = (1-2\alpha)^{-1} + \tfrac{1}{2}\tilde{q}^\pm + O(1-2\alpha),$$

or

$$q^+ = -1 + (1-2\alpha)\tilde{q}^+ + O[(1-2\alpha)^2], \qquad q^- = -1 + (1-2\alpha)(\tilde{q}^- + \mathrm{i}\pi) + O[(1-2\alpha)^2].$$

It is evident that when $\alpha \to \frac{1}{2}$

$$q^+ - q^- = (1-2\alpha)(\tilde{q}^+ - \tilde{q}^- - \mathrm{i}\pi) + O[(1-2\alpha)^2]$$

necessarily tends to zero, but $\Delta = \tilde{q}^+ - \tilde{q}^- - \mathrm{i}\pi \neq 0$, so that

$$C(\tau) = \frac{2(M^+ - M^-)}{\tilde{q}^+ - \tilde{q}^- - \mathrm{i}\pi}$$

(cf. (3.21) in Paper I).

(*b*) The singularity in (3.15) also appears if $d_1^+ = d_1^- = 0$ (then $q^+ = q^- = 0$) or $d_2^+ = d_2^- = 0$ (then $q^+ - q^-$ is not defined). In either case the second harmonic itself is a neutral mode, and the main nonlinear process will be not the wave self-action considered in this paper but the resonant interaction of the fundamental and the second harmonic (or, in more conventional terminology, the harmonic–subharmonic interaction, see, for example, Collins & Maslowe 1988). Note that if the stability boundary is as simply structured as in the Drazin, Holmboe and (2.7*a*, *b*) models, then α varies along it monotonically, and only the case $d_1^+ = d_1^- = 0$ ($q^+ = q^- = 0$) is possible. This is realized in Drazin's model when $\omega_N^2 = \frac{1}{5}$, $J_N = \frac{4}{25}$ and in Holmboe's model when $\omega_N = \frac{1}{3}$ and $J_N = \frac{2}{9}$, but it is not realized in (2.7*a*, *b*) when $D \neq 0$ because of the dependence of the phase velocity c on ω.

(*c*) Finally, one of the coefficients $d_2^{\pm}$, for example, d_2^+, can become zero. In the limit $d_2^+ \to 0$ we obtain $q^+ \to \infty$, $b_2^{\pm} = 0$, so that $M^- = 0$, $C(\tau) = 0$ and $\Phi_2^{(3)} = 0$, but $M^+ = b_2^+ q^+$ remains finite and $\Psi_2^{(3)}$ is matched to the outer solution at an arbitrary $(M^+ - M^-)$ jump without generating Φ. It may be that $\Phi \neq 0$ would appear on the main sequence later, when matching to the outer solution, say, $O(\varepsilon^4 \mu^{4\alpha-8})$ of the third harmonic.

Appendix D. On the solutions of nonlinear evolution equations in the unsteady CL regime

D.1. *Cubic nonlinearity*

Consider an NEE with a cubic nonlinearity of the form (cf. (3.25) and (3.26))

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + b\mathrm{e}^{-\mathrm{i}\chi} \int_0^\infty \mathrm{d}s\, s^{\lambda-1} \int_0^1 \mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma) A(x-s) A(x-\sigma s) \overline{A}(x-s-\sigma s), \quad \text{(D 1)}$$

in which $b > 0$, $\lambda > 0$, $\mu > 0$, $|\chi| \leqslant \pi$, the kernel $G(\sigma)$ is real, with $G(0) = g_0 \neq 0$, and the integral

$$I_0 \equiv \int_0^\infty \frac{\mathrm{d}s\, s^{\lambda-1}}{(1+s)^{(\lambda+1)/2}} \int_0^1 \frac{\mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma)}{[(1+\sigma s)(1+s+\sigma s)]^{(\lambda+1)/2}} > 0. \quad \text{(D 2)}$$

For the numerical solution with the 'initial' condition (4.1), it is convenient to recast (D 1) in a 'universal' form by passing to a normalized amplitude C:

$$\left.\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}T} &= \mathrm{e}^{-\mathrm{i}\chi} \int_0^\infty \mathrm{d}t\, t^{\lambda-1} \mathrm{e}^{-t} \int_0^1 \frac{\mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma)}{(1+\sigma)^\lambda} C(T\mathrm{e}^{-t/(1+\sigma)}) C(T\mathrm{e}^{-\sigma t/(1+\sigma)}) \overline{C}(T\mathrm{e}^{-t}), \\ C(0) &= 1, \end{aligned}\right\} \quad \text{(D 3)}$$

where

$$A(x) = A_0 C(T) \mathrm{e}^{\gamma_L x}, \quad T = \frac{bA_0^2}{(2\gamma_L)^{\lambda+1}} \mathrm{e}^{2\gamma_L x}, \quad t = 2\gamma_L(1+\sigma)s.$$

Let us note in passing that because of a rapid (explosive) growth of the solutions of (D 3), when representing them in graphical form it is convenient to introduce a somewhat different amplitude function

$$\tilde{C}(T) = \frac{C(T)\ln(1+|C(T)|)}{|C(T)|\ln 2},$$

which is used in figures 11(*b–e*).

At the nonlinear stage of evolution, the linear term on the right-hand side of (D 1) becomes small by comparison with the nonlinear term, and it can be neglected by formally putting $\gamma_L = 0$. In this case (D 1) has an exact analytic solution of the explosive form

$$A = B(x_0 - x)^{-a+\mathrm{i}\beta}, \qquad a = \frac{\lambda+1}{2}, \tag{D 4}$$

in which the constant β and B are determined from the equation

$$a - \mathrm{i}\beta = b|B|^2 \mathrm{e}^{-\mathrm{i}\chi} \int_0^\infty \frac{\mathrm{d}s\, s^{\lambda-1}}{(1+s)^a} \int_0^1 \frac{\mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma)}{[(1+\sigma s)(1+s+\sigma s)]^a} \exp\left[\mathrm{i}\beta \ln \frac{(1+s)(1+\sigma s)}{1+s+\sigma s}\right]. \tag{D 5}$$

In view of the inequality (D 2) when $\chi = 0$ equation (D 5) has the solution $\beta = 0$. (If, however, $I_0 < 0$, then $\beta = 0$ at $\chi = \pm\pi$.) By and large $\beta(\chi)$ is an odd function: if $\beta(\chi) = f(\chi)$ is the solution of (D 5), then $\beta(-\chi) = -f(\chi)$ is also the solution. When $\beta(\chi)$ is considered on the entire real axis χ, it becomes evident that $\beta = f(\chi + 2\pi n)$ at any integer n will also be the solution of (D 5). In particular, $\beta(2\pi n) = 0$, and $\beta(\chi)$ does not have any other zeros. The function that has such properties is normally multiple-valued, in some domains of χ at least. In fact, if (D 5) when $\chi = \pi$ has the solution $\beta = \beta_1$ (note that $\beta_1 \neq 0$), then it also has the solution $\beta = -\beta_1$, and $\beta(\chi)$ here is two-valued at least (and in the general case it can also take several pairs of values, $\pm\beta_i$).

To study $\beta(\chi)$, it is convenient to transform (D 5). Multiplying by $(a + \mathrm{i}\beta)$ and separating the real and imaginary parts yields

$$I_1(\beta,\chi) \equiv \int_0^\infty \frac{\mathrm{d}s\, s^{\lambda-1}}{(1+s)^a} \int_0^1 \frac{\mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma)}{[(1+\sigma s)(1+s+\sigma s)]^a} \sin\left[\beta \ln\left(1 + \frac{\sigma s^2}{1+s+\sigma s}\right) + \chi_0 - \chi\right] = 0,$$

$$I_2(\beta,\chi) \equiv \int_0^\infty \frac{\mathrm{d}s\, s^{\lambda-1}}{(1+s)^a} \int_0^1 \frac{\mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma)}{[(1+\sigma s)(1+s+\sigma s)]^a} \cos\left[\beta \ln\left(1 + \frac{\sigma s^2}{1+s+\sigma s}\right) + \chi_0 - \chi\right] > 0,$$

$$|B|^2 = \frac{(a^2+\beta^2)^{1/2}}{b I_2(\beta,\chi)}, \qquad \chi_0 = \arctan\frac{\beta}{a}; \qquad a = \frac{\lambda+1}{2}. \tag{D 6}$$

Upon calculating the derivative

$$\frac{\mathrm{d}\beta}{\mathrm{d}\chi} = \frac{I_2(\beta,\chi)}{I_3(\beta,\chi)}, \tag{D 7}$$

$$I_3(\beta,\chi) = \int_0^\infty \frac{\mathrm{d}s\, s^{\lambda-1}}{(1+s)^a} \int_0^1 \frac{\mathrm{d}\sigma\, \sigma^{\mu-1} G(\sigma)}{[(1+\sigma s)(1+s+\sigma s)]^a} \left[\ln\left(1 + \frac{\sigma s^2}{1+s+\sigma s}\right) + \frac{a}{a^2+\beta^2}\right]$$
$$\times \cos\left[\beta \ln\left(1 + \frac{\sigma s^2}{1+s+\sigma s}\right) + \chi_0 - \chi\right],$$

one can see that each branch of $\beta(\chi)$ is monotonic, and it starts and ends at singular points where either $I_2 = 0$ or $I_3 = 0$ (or both).

We restrict the discussion to the simplest case where at the ends of each branch $|\beta| \to \infty$ (the author is unaware of any other cases in specific evolution problems). The main branch (that passes through the origin of the coordinates) represents comprehensively all properties of the complete solution of (D 5) because other branches are obtained from this branch through a shift in χ by 2π. Moreover, in view of the oddness of $\beta(\chi)$ it will suffice to consider $\beta \geqslant 0$ only.

When $\beta \to +\infty$ the integral $I(\beta,\chi) = I_2 + \mathrm{i}I_1$ tends to zero, the region $\sigma s^2 = O(\beta^{-1})$ makes the main contribution to it, and its order depends on the parameters λ and μ. Straightforward but somewhat unwieldy calculations give

$$I(\beta,\chi) = \beta^{-\lambda/2}\frac{\Gamma(\frac{1}{2}\lambda)}{2}\int_0^1 \mathrm{d}\sigma\, \sigma^{\mu-\lambda/2-1}G(\sigma)\exp\left\{\mathrm{i}\left[\frac{\pi}{2}\left(\frac{\lambda}{2}+1\right)-\chi\right]\right\}+O(\beta^{-\mu}+\beta^{-(\lambda+1)/2}) \qquad \text{(D 8}a\text{)}$$

when $\lambda < 2\mu$,

$$I(\beta,\chi) = \beta^{-\mu}\frac{\mu[\Gamma(\mu)]^2\Gamma(\lambda-2\mu)}{\Gamma(\lambda-\mu+1)}g_0\exp\left\{\mathrm{i}\left[\frac{\pi}{2}(\mu+1)-\chi\right]\right\} + O(\beta^{-\lambda/2}+\beta^{-\mu-1}) \qquad \text{(D 8}b\text{)}$$

when $\lambda > 2\mu$, and

$$I(\beta,\chi) = \beta^{-\mu}\frac{\Gamma(\mu)}{2}\left\{g_0\left[\ln\beta - 2C - 3\psi(\mu) - \frac{2}{\mu} - \frac{\mathrm{i}\pi}{2}\right] - \int_0^1 \mathrm{d}\sigma\frac{\mathrm{d}G(\sigma)}{\mathrm{d}\sigma}\ln\sigma\right\}$$
$$\times\exp\left\{\mathrm{i}\left[\tfrac{1}{2}\pi(\mu+1)-\chi\right]\right\} + O(\beta^{-\mu-1/2}) \qquad \text{(D 8}c\text{)}$$

when $\lambda = 2\mu$. Here $C = 0.577216$ is the Euler's constant and $\psi(z) = \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$. Note that $\arg I(\beta,\chi)$ is defined not only by the expression appearing in the corresponding exponential, but also by the sign of

$$I_{00} = \int_0^1 \mathrm{d}\sigma\sigma^{\mu-\lambda/2-1}G(\sigma)$$

in (D 8a) and of g_0 in (D 8b,c). Considering that along the branch $\beta = \beta(\chi)$ the integral $I(\beta,\chi)$ must remain real and positive, we find that $\beta \to +\infty$ when $\chi \to \chi_+ + 2\pi n - 0$, where

$$\chi_+ = \begin{cases} \frac{1}{2}\pi(\mu + \operatorname{sgn} g_0), & \lambda \geqslant 2\mu, \\ \frac{1}{2}\pi\left(\frac{1}{2}\lambda + \operatorname{sgn} I_{00}\right), & \lambda < 2\mu. \end{cases} \qquad \text{(D 9)}$$

D.2. *Quintic nonlinearity*

Consider NEE of the form (cf. (3.27))

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + b\mathrm{e}^{-\mathrm{i}\chi}\int_0^\infty \mathrm{d}t\, t^{\lambda-1}\int_0^\infty \mathrm{d}s\, s^{\mu-1}\int_0^1 \mathrm{d}u\, u^{\nu-1}\int_0^1 \mathrm{d}v\, G(u,v)A(x-t-s)$$
$$\times A(x-t-us)A(x-t-uvs)\overline{A}(x-t-(1+u+uv)s)\overline{A}(x-2t), \qquad \text{(D 10)}$$

in which $b > 0$, $\lambda > 0$, $\mu > 0$, $\nu > 0$, $|\chi| \leqslant \pi$, the kernel $G(u,v)$ is real, with $G(0,v) \not\equiv 0$, and the integral

$$I_0 = \int_0^\infty \frac{\mathrm{d}s\, s^{\mu-1}}{(1+s)^a}\int_0^1 \mathrm{d}u\, u^{\nu-1}\int_0^1 \frac{\mathrm{d}v\, G(u,v)}{[(1+us)(1+uvs)(1+s+us+uvs)]^a} > 0, \qquad \text{(D 11)}$$

where $a = (\lambda+\mu+1)/4$ is the exponent in the law of explosive growth (4.2) (cf. (D 4)).

For the numerical solution with the 'initial' condition (4.1), NEE (D 10) is also conveniently recast in a 'universal' form:

$$\frac{\mathrm{d}C}{\mathrm{d}T} = T\mathrm{e}^{-\mathrm{i}\chi}\int_0^\infty \mathrm{d}t\, t^{\lambda-1}\mathrm{e}^{-t}\int_0^\infty \mathrm{d}s\, s^{\mu-1}\mathrm{e}^{-s}\int_0^1 \mathrm{d}u\, u^{\nu-1}\int_0^1 \frac{\mathrm{d}v\, G(u,v)}{(1+u+uv)^{\mu}} C\left(T\mathrm{e}^{-t_1-s_1}\right)$$
$$\times C\left(T\mathrm{e}^{-t_1-us_1}\right) C\left(T\mathrm{e}^{-t_1-uvs_1}\right) \overline{C}\left(T\mathrm{e}^{-t_1-s}\right) \overline{C}\left(T\mathrm{e}^{-2t_1}\right),$$

$$C(0) = 1,$$

where

$$A(x) = A_0 C(T)\mathrm{e}^{\gamma_L x}, \quad T = \frac{b^{1/2}A_0^2}{3^{\lambda/2}(2\gamma_L)^{2a}}\mathrm{e}^{2\gamma_L x}, \quad t_1 = \frac{t}{3}, \quad s_1 = \frac{s}{1+u+uv}.$$

The function $\beta(\chi)$ has the same properties (multivaluedness, oddness, and 2π-periodicity) as in the case of a cubic nonlinearity. For determining it, it is necessary to solve the system (cf. (D 6)):

$$I_1 \equiv \int_0^\infty \frac{\mathrm{d}s\, s^{\mu-1}}{(1+s)^a}\int_0^1 \mathrm{d}u\, u^{\nu-1}\int_0^1 \frac{\mathrm{d}v\, G(u,v)\sin\left[\beta\ln\dfrac{(1+s)(1+us)(1+uvs)}{1+s+us+uvs}+\chi_0-\chi\right]}{[(1+us)(1+uvs)(1+s+us+uvs)]^a} = 0,$$

$$I_2 \equiv \int_0^\infty \frac{\mathrm{d}s\, s^{\mu-1}}{(1+s)^a}\int_0^1 \mathrm{d}u\, u^{\nu-1}\int_0^1 \frac{\mathrm{d}v\, G(u,v)\cos\left[\beta\ln\dfrac{(1+s)(1+us)(1+uvs)}{1+s+us+uvs}+\chi_0-\chi\right]}{[(1+us)(1+uvs)(1+s+us+uvs)]^a} > 0,$$

$$\chi_0 = \arg J_0 + \arctan\frac{\beta}{a}; \quad J_0 = \int_0^1 \frac{\mathrm{d}s\, s^{\lambda-1}(1-s)^{a-\mathrm{i}\beta}}{(1+s)^{a+\mathrm{i}\beta}}. \tag{D 12}$$

Proceeding in the same manner as in the case of cubic nonlinearity one can show that $\beta \to +\infty$ when $\chi \to \chi_+ + 2\pi n$, where

$$\chi_+ = \begin{cases} \frac{1}{2}\pi\left(\frac{1}{2}\lambda + \nu + \mathrm{sgn}J_1\right), & \mu \geqslant 2\nu, \\ \frac{1}{2}\pi\left(\frac{1}{2}(\lambda+\mu) + \mathrm{sgn}J_2\right), & \mu < 2\nu. \end{cases} \tag{D 13}$$

$$J_1 = \int_0^1 \frac{\mathrm{d}v\, G(0,v)}{(1+v)^{\nu}}, \quad J_2 = \int_0^1 \mathrm{d}u\, u^{\nu-\mu/2-1}\int_0^1 \frac{\mathrm{d}v\, G(u,v)}{(1+v+uv)^{\mu/2}}.$$

For the nonlinearities considered in this paper, using (D 9) and (D 13) we find:
in NEE (3.25) $\lambda = 2\mu = 8-4\alpha$ and $g_0 < 0$, hence $\chi_+ = (3/2-\alpha)\pi$;
in NEE (3.26) $\lambda = 4-4\alpha$, $\mu = 3-2\alpha$ and $I_{00} > 0$, hence $\chi_+ = (3/2-\alpha)\pi$;
in NEE (3.27) $\lambda = 2-\alpha$, $\mu = 6-3\alpha$, $\nu = 5-\alpha$ and $J_2 > 0$, hence $\chi_+ = (5/2-\alpha)\pi$.
If the nonlinearity is non-dissipative (NEE (3.26) or (3.27)), the numerically inferred dependence $\beta(\chi)$ (see figure 10*b,c*) accords well with the theory outlined above, and the main branch tends to $+\infty$ when $\chi \to \chi_+ - 0$.

In the case of the dissipative nonlinearity, the picture is more complicated. When $\alpha = \alpha_* = 0.615$, the integral involved in (D 2) becomes zero. The inequality (D 2) could be satisfied through a change of the common sign of the kernel when passing through $\alpha = \alpha_*$; in this case, however, the α-dependence of both the kernel itself and the phase χ ceases to be continuous. We chose to retain the continuous dependence on α, and we defined the kernel $G_\nu(\sigma)$ such that $I_0 > 0$ when $0 \leqslant \alpha < \alpha_*$ and $I_0 < 0$

when $\alpha_* < \alpha < 1$. This being so, on the main branch $\beta(\chi) = 0$ when $\chi = 0$ if $\alpha < \alpha_*$ and when $\chi = \pi$ if $\alpha > \alpha_*$. Furthermore, at any α the main branch tends to $+\infty$ not when $\chi \to \chi_+$ but when $\chi \to \chi_+ + 2\pi$. These characteristics are all evident in figure 10(a).

REFERENCES

Abramowitz, M. & Stegun, I. A. 1964 *Handbook of Mathematical Functions*. National Bureau of Standards.

Andronov, A. A. & Fabrikant, A. L. 1979 Landau damping, wind waves and whistle. In *Nonlinear Waves* (ed. A. V. Gaponov-Grekhov), pp. 68–104. Nauka (in Russian).

Brown, S. N., Rosen, A. S. & Maslowe, S. A. 1981 The evolution of a quasi-steady layer in a stratified viscous shear layer. *Proc. R. Soc. Lond.* A **375**, 271–293.

Brown, S. N. & Stewartson, K. 1978 The evolution of a small inviscid disturbance to a marginally unstable stratified shear flow: stage two. *Proc. R. Soc. Lond.* A **363**, 175–194.

Churilov, S. M. 1989 The nonlinear stabilization of a zonal shear flow instability. *Geophys. Astrophys. Fluid Dyn.* **46**, 159–175.

Churilov, S. M. & Shukhman, I. G. 1987 Nonlinear stability of a stratified shear flow: viscous critical layer. *J. Fluid Mech.* **180**, 1–20.

Churilov, S. M. & Shukhman, I. G. 1988 Nonlinear stability of a stratified shear flow in the regime with an unsteady critical layer. *J. Fluid Mech.* **194**, 187–216 (Paper I).

Churilov, S. M. & Shukhman, I. G. 1992 Critical layer and nonlinear evolution of disturbances in weakly supercritical shear layer. In *XVIIIth Intl Congress of Theor. and Appl. Mech., Haifa, Israel.* Abstracts, pp. 39–40. Also: *Atmos. Oceanic Phys.* **31**, 534–546 (1996).

Churilov, S. M. & Shukhman, I. G. 1994 Nonlinear spatial evolution of helical disturbances to an axial jet. *J. Fluid Mech.* **281**, 371–402.

Churilov, S. M. & Shukhman, I. G. 1996 The nonlinear critical layer resulting from the spatial or temporal evolution of weakly unstable disturbances in shear flows. *J. Fluid Mech.* **318**, 189–221.

Collins, D. A. & Maslowe, S. A. 1988 Vortex pairing and resonant wave interactions in a stratified free shear layer. *J. Fluid Mech.* **191**, 465–480.

Drazin, P. G. 1958 The stability of a shear layer in an unbounded heterogeneous inviscid fluid. *J. Fluid Mech.* **4**, 214–224.

Drazin, P. G. & Reid, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

Erdelyi, A. 1953 *Higher Transcendental Functions*, vol. 1. McGraw-Hill.

Goldstein, M. E. & Choi, S.-W. 1989 Nonlinear evolution of interacting oblique waves on two dimensional shear layers. *J. Fluid Mech.* **207**, 97–120. Also Corrigendum, *J. Fluid Mech.* **216**, 659–663.

Goldstein, M. E. & Hultgren, L. S. 1988 Nonlinear spatial evolution of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **197**, 295–330. Also Corrigendum, *J. Fluid Mech.* **281**, 403–404.

Goldstein, M. E. & Leib, S. J. 1989 Nonlinear evolution of oblique waves on compressible shear layer. *J. Fluid Mech.* **207**, 73–96.

Hazel, P. 1972 Numerical studies of the stability of inviscid stratified shear flows. *J. Fluid Mech.* **51**, 39–61.

Hickernell, F. J. 1984 Time-dependent critical layers in shear flows on a beta-plane. *J. Fluid Mech.* **142**, 431–449.

Holmboe, J. 1960 Unpublished Lectures Notes. University of California, Los Angeles.

Hultgren, L. S. 1992 Nonlinear spatial equilibration of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **236**, 635–664.

Lott, F. & Teitelbaum, H. 1992 Nonlinear dissipative critical level interaction in a stratified shear flow: instabilities and gravity waves. *Geophys. Astrophys. Fluid Dyn.* **66**, 133–167.

Miles, J. W. 1961 On the stability of heterogeneous shear flows. *J. Fluid Mech.* **10**, 496–508.

Miles, J. W. 1963 On the stability of heterogeneous shear flows. Part 2. *J. Fluid Mech.* **16**, 209–227.

Shukhman, I. G. 1989 Nonlinear stability of a weakly supercritical mixing layer in a rotating fluid. *J. Fluid Mech.* **200**, 425–450.

Shukhman, I. G. 1991 Nonlinear evolution of spiral density waves generated by the instability of the shear layer in a rotating compressible fluid. *J. Fluid Mech.* **233**, 587–612.

Shukhman, I. G. & Churilov, S. M. 1997 Effect of slight stratification on the nonlinear spatial evolution of a weakly unstable wave in a free shear layer. *J. Fluid Mech.* **343**, 197–233.

Thorpe, S. A. 1969 Neutral eigensolutions of the stability equation for stratified shear flow. *J. Fluid Mech.* **36**, 673–683.

Timofeev, A. V. 1970 The oscillations of the plasma and liquid inhomogeneous flows. *Uspekhi Fiz. Nauk*, **102**, 185–210 (in Russian).

Timofeev, A. V. 1989 Resonant effects in oscillations of inhomogeneous flows of continuous media. In *Voprosy Teorii Plasmy (Reviews of Plasma Physics)* (ed. B. B. Kadomtsev), vol. 17, pp. 157–244. Energoatomizdat (in Russian).

Wu, X., Lee, S. S. & Cowley, S. J. 1993 On the weakly nonlinear three-dimensional instability of shear layers to pairs of oblique waves: the Stokes layer as a paradigm. *J. Fluid Mech.* **253**, 681–721.